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# VIII.—*On the Equilibrium and Motion of Solid and Fluid Bodies.* *By the* Rev. Samuel Haughton, F. T. C. D.

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THE object of the present paper is to deduce from simple physical considerations the laws of equilibrium and motion of elastic solid and fluid bodies, by the method followed in the *Mecanique Analytique* of Lagrange, which possesses the remarkable advantage of giving, by the same analysis, the *general* equations of any system, together with the *particular* conditions to be fulfilled at the limits. This method is particularly valuable in such cases as the present, where the problem is, to determine the conditions of equilibrium or motion of an indefinite number of material points, situated indefinitely near each other, and acting according to certain laws; although it may be admitted that the ordinary methods usual in mechanics are preferable to Lagrange's method, in cases where the points of application of the forces of the system are definite in number, and situated at finite distances from each other. The first application of Lagrange's method to the problem of elastic solids was made by Navier (*Mem. de l'Institut.* Tom. VII.), who discussed the laws of *equilibrium* of a *homogeneous* uncrystalline solid, but does not appear to have ever undertaken the general problem, or to have obtained any *dynamical* results. The present paper is an attempt to apply the same method of Lagrange to the general case of material substances, whether fluid or solid, homogeneous or heterogeneous, and whether possessed of a crystalline structure or not; and more particularly to investigate the general dynamical laws of solid elastic bodies, and the conditions at the limiting surfaces which bound the solid.

In the method of Lagrange the forces in action in such a system as a continuous body must be divided into two parts—the external and the internal forces of the system; and the general equation of equilibrium is determined by expressing that the sum of the *moments* (as defined by Lagrange) of the external forces is equal to the sum of the *moments* of the internal forces; hence the general equation of the system is

$$\iiint (\mathrm{X}\delta\xi + \mathrm{Y}\delta\eta + \mathrm{Z}\delta\zeta)dm = \iiint \delta\mathrm{V}\,dxdydz, \qquad (1)$$

where $x, y, z$ are the coordinates of the position of rest of any molecule $(m)$, and $x+\xi, y+\eta, z+\zeta$, are the coordinates of the same molecule when displaced; and the function V depends on the medium in such a manner that $\delta\mathrm{V}$ is the sum of the moments of internal forces acting at any point of the medium; and $\iiint \delta\mathrm{V}\,dxdydz$ represents the sum of all the moments arising from internal forces. It appears, therefore, that in investigating the laws of any medium, it is necessary to know the form of the function V; being given the form of V, the laws of the medium and of the actions of its molecules on each other are known, and *vice versâ*, if we know the nature of the medium, we can determine the form of V. Either of two methods may, therefore, be followed, in discussing the conditions of a system such as a solid or fluid body; the form of the function V may be assumed, and the truth of the assumption afterwards verified,—or the form of the function V may be determined from whatever assumption we make, as to the mode of action of the molecules of the system on each other: the latter method is used in this paper. The assumed definition of solid and fluid bodies is the following: that they *consist of molecules, which act on each other only in the line joining them;* i. e. the whole of the action of one molecule on another is a force in the line joining the molecules, and none at all in a direction perpendicular to that line. This force is also assumed to be a function of the distance of the molecules, and in the case of crystalline structure also of the direction of the line joining them. All solid and fluid bodies are either homogeneous or heterogeneous in their structure: in homogeneous bodies the molecular forces will, *ceteris paribus*, be the same at all points of the body, but in heterogeneous bodies they will vary from point to point of the body, and become functions of $(x, y, z)$. A crystalline structure is different, and consists in this, that the action of the molecules surrounding any molecule $(m)$, depends upon the *direction* in which they lie, and not merely on their distance from $(m)$. In order to avoid misconception, it is necessary to state that the *molecular* forces alone are included in the internal forces; the mutual *gravitations* of the particles towards each other, although they might with propriety be called internal forces, are here included among the external forces, as they are quite different from the molecular forces, and act at finite distances; and any statement made in reference to the function V is to be understood only of the molecular forces, which alone are included in it. Also, all such forces as those arising from the pressures of the sides of vessels containing a fluid, &c. are excluded from the function V.

Let $(\rho, \alpha, \beta, \gamma)$ be the original length and direction of the line $(mm')$ joining two molecules; suppose these to become $(\rho + \rho', \alpha + \alpha', \beta + \beta', \gamma + \gamma')$, where $(\rho', \alpha', \beta', \gamma')$ are the changes in length and direction; it is evident that the force acting between $m$ and $m'$ will be, *in general*, a function of $(\rho, \alpha, \beta, \gamma, \rho', \alpha', \beta', \gamma')$, and also of $(x, y, z)$; or, if $(f)$ denote this force, it may be represented by

$$f = F_0 + 2F_1\rho' + 3F_2\rho'^2 + \text{\&c.}$$
$$+ \left\{ \begin{array}{l} A_1\alpha' + A_2\alpha'^2 + \text{\&c.} \\ B_1\beta' + B_2\beta'^2 + \text{\&c.} \\ C_1\gamma' + C_2\gamma'^2 + \text{\&c.} \end{array} \right\} \tag{2}$$

$F_0$, $F_1$, &c., $A_1$, $B_1$, $C_1$, &c., being functions of $(\rho, \alpha, \beta, \gamma, x, y, z)$ in general; and the part of $(f)$ depending on $(\rho')$ arises from the change of *distance;* while the part depending on $(\alpha', \beta', \gamma')$ arises from the change of *direction.* If we consider, however, that the change of force produced by a change in direction is *continuous,* while an indefinitely small change of distance developes a *finite* molecular force, it will be evident that we may neglect the part of $(f)$ depending on change of direction, together with the part arising from high powers of $(\rho')$; consequently

$$f = F_0 + 2F_1\rho' + \text{\&c.}, \tag{3}$$

where $F_0$, $F_1$, &c., are functions of $(\rho, \alpha, \beta, \gamma)$, and implicitly of $(x, y, z)$, if external forces act on the system, or if the body be naturally heterogeneous.

As the force $(f)$ is a force tending to alter $\rho + \rho'$, its moment will be $f.\delta\rho'$; hence, neglecting the higher powers of $\rho'$, we shall have the sum of moments of internal forces at a given point,

$$\delta V = \Sigma(F_0\delta\rho' + F_1\delta(\rho')^2);$$

and, finally,

$$V = \int_0^\infty\int_0^\pi\int_0^{2\pi}(F_0\rho' + F_1\rho'^2)\,\rho^2\sin\theta\,d\rho\,d\theta\,d\phi, \tag{4}$$

and

$$V = V_0 + V_1;$$

$V_0$ arising from the terms $F_0$, and $V_1$ arising from the terms $F_1$.

This value of the function $V$ contains all solid and fluid bodies, together with the class of bodies which cannot be said to be either solid or perfectly fluid, and which are generally called "viscous" fluids. If we knew sufficiently well the difference of molecular structure of these different classes of bodies, we should be able to determine the proper $V$ for each of them; but in the absence of accurate knowledge on this point, we must only give the best definition we can of solids and fluids, and the agreement of the *secondary laws* deduced from the definition, with the observed laws of these bodies, will be sufficient proof of the correctness of the definition founded on their molecular structure.

I define a solid body to be of such a nature that the part of the molecular force depending on $F_0$ does not exist, so that in a solid body, $F_0$ and, consequently, $V_0$, is zero. In all other bodies, liquid, gaseous, and viscous, the function $V$ will contain a part $V_0$, which does not enter into the function peculiar to solid bodies. The definition of a solid by means of the assumption that $F_0$ and $V_0$ are zero, is the same thing as supposing that in a solid body the molecular forces equilibrate each other, without the aid of external forces; while in fluid bodies the molecular forces do not equilibrate themselves without the aid of external forces and pressures; and this notion of a solid seems to be correct from the nature of the molecular forces themselves, which appear to be attractive up to a certain distance; but if the distance be still further diminished, the force becomes repulsive. The condition $V_0 = 0$, may then arise from the molecules of a solid body being placed at such distances from each other that they are held in equilibrium by the action of powerful, attractive, and repulsive forces acting in opposite directions. However, our ignorance of the internal structure of bodies is so great, that such considerations as these would be, by themselves, an unsafe foundation for a theory of solids and fluids; and the best course is, to use them only as indications of the correct definitions, which must themselves be ascertained by the agreement of the results they lead to, with the known laws of solid and fluid bodies.

In following this method in the present paper, I hope to show that the presence of $V_0$ will produce the equations of hydrostatics and hydrodynamics, while the term $V_1$ will give all the laws of solid bodies; so that we have the following functions;—for solids, $V = V_1$;

for perfect fluids, $V = V_0$;

for viscous fluids, $V = V_0 + V_1$.

In perfect fluids the terms arising from $\mathrm{V}_1$ may be neglected in comparison with those produced by $\mathrm{V}_0$, at least in hydrostatics; though I think it probable that in hydrodynamics the use of the function $\mathrm{V}_0 + \mathrm{V}_1$ would be sometimes necessary. Also in viscous fluids $\mathrm{V}_0$, and the terms produced by it, will be very small, so that in this case we should always be compelled to use the function $\mathrm{V}_0 + \mathrm{V}_1$.

I now proceed to the investigation of the equations of equilibrium and motion of fluids, or bodies in which $\mathrm{V} = \mathrm{V}_0$. The general equation of equilibrium and motion in this case will be

$$\iiint (\mathrm{X}\delta\xi + \mathrm{Y}\delta\eta + \mathrm{Z}\delta\zeta)\, dm = \iiint \delta\mathrm{V}_0\, dx\, dy\, dz, \tag{5}$$

where

$$\mathrm{V}_0 = \int_0^\infty\int_0^\pi\int_0^{2\pi} \mathrm{F}_0\rho'.\rho^2 \sin\theta\, d\rho\, d\theta\, d\phi.$$

The value of $\rho'$, in terms of $\xi$, $\eta$, $\zeta$, &c., to be substituted in this formula, is thus found. Let $x$, $y$, $z$, $x+a$, $y+b$, $z+c$, be the coordinates of the positions of rest of $m$ and $m'$; then in the altered position of the molecules, if $x, y, z$ become $x+\xi$, $y+\eta$, $z+\zeta$, the coordinates of $m'$ will become

$$x + \xi + a + \frac{d\xi}{dx}a + \frac{d\xi}{dy}b + \frac{d\xi}{dz}c,$$

$$y + \eta + b + \frac{d\eta}{dx}a + \frac{d\eta}{dy}b + \frac{d\eta}{dz}c,$$

$$z + \zeta + c + \frac{d\zeta}{dx}a + \frac{d\zeta}{dy}b + \frac{d\zeta}{dz}c,$$

and consequently $\rho + \rho'$, being equal to the square root of the sum of squares of differences of coordinates of its extreme points, will be expressed by the formula,

$$\rho+\rho' = \sqrt{\left\{\left(a + \frac{d\xi}{dx}a + \frac{d\xi}{dy}b + \frac{d\xi}{dz}c\right)^2 + \left(b + \frac{d\eta}{dx}a + \frac{d\eta}{dy}b + \frac{d\eta}{dz}c\right)^2 + \left(c + \frac{d\zeta}{dx}a + \frac{d\zeta}{dy}b + \frac{d\zeta}{dz}c\right)^2\right\}};$$

from which, by neglecting the smaller quantities, and assuming

$$\mathrm{U} = \frac{d\eta}{dz} + \frac{d\zeta}{dy}, \quad \mathrm{V} = \frac{d\zeta}{dx} + \frac{d\xi}{dz}, \quad \mathrm{W} = \frac{d\xi}{dy} + \frac{d\eta}{dx},$$

we obtain

$$\rho + \rho' = \rho + \rho\left(\frac{d\xi}{dx}\cos^2 a + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \mathrm{U}\cos\beta\cos\gamma + \mathrm{V}\cos a\cos\gamma + \mathrm{W}\cos a\cos\beta\right),$$

and, finally,

$$\rho' = \rho\left(\frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \mathrm{U}\cos\beta\cos\gamma + \mathrm{V}\cos\alpha\cos\gamma + \mathrm{W}\cos\alpha\cos\beta\right). \quad (6)$$

Substituting this value of $\rho'$ in (5), we obtain

$$\mathrm{V}_0 = \int_0^\infty\int_0^\pi\int_0^{2\pi} \mathrm{F}_0\left(\frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \mathrm{U}\cos\beta\cos\gamma + \mathrm{V}\cos\alpha\cos\gamma + \mathrm{W}\cos\alpha\cos\beta\right)\rho^3\sin\theta\, d\rho\, d\theta\, d\phi;$$

or, since,

$$\cos\alpha = \cos\phi\sin\theta,\ \cos\beta = \sin\phi\sin\theta,\ \cos\gamma = \cos\theta,$$

$$\mathrm{V}_0 = \mathrm{A}_1\frac{d\xi}{dx} + \mathrm{B}_1\frac{d\eta}{dy} + \mathrm{C}_1\frac{d\zeta}{dz} + \mathrm{D}_1\mathrm{U} + \mathrm{E}_1\mathrm{V} + \mathrm{F}_1\mathrm{W},$$

i. e. $\mathrm{V}_0$ is a *homogeneous function of the first order of the six quantities,*

$$\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}, \mathrm{U}, \mathrm{V}, \mathrm{W},$$

where

$$\mathrm{A}_1 = \iiint \mathrm{F}_0.\cos^2\phi\sin^3\theta\rho^3 d\rho d\theta d\phi, \qquad \mathrm{D}_1 = \iiint \mathrm{F}_0.\sin\phi\sin^2\theta\cos\theta\rho^3 d\rho d\theta d\phi,$$
$$\mathrm{B}_1 = \iiint \mathrm{F}_0.\sin^2\phi\sin^3\theta\rho^3 d\rho d\theta d\phi, \qquad \mathrm{E}_1 = \iiint \mathrm{F}_0.\cos\phi\sin^2\theta\cos\theta\rho^3 d\rho d\theta d\phi,$$
$$\mathrm{C}_1 = \iiint \mathrm{F}_0.\cos^2\theta\sin\theta.\rho^3 d\rho d\theta d\phi, \qquad \mathrm{F}_1 = \iiint \mathrm{F}_0.\sin\phi\cos\phi.\sin^3\theta\rho^3 d\rho d\theta d\phi;$$

but, since in fluids, the structure is not crystalline, $\mathrm{F}_0$ will not be a function of $(\theta, \phi)$, but only of $\rho$; and by integrating twice with respect to $\theta$ and $\phi$, we obtain,

$$\mathrm{D}_1 = \mathrm{E}_1 = \mathrm{F}_1 = 0,$$
$$\mathrm{A}_1 = \mathrm{B}_1 = \mathrm{C}_1 = p = \frac{4\pi}{3}\int_0^\infty \mathrm{F}_0\rho^3 d\rho;$$

hence we deduce, finally, for the value of $\mathrm{V}_0$, the expression

$$\mathrm{V}_0 = p\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right). \quad (7)$$

Substituting this expression for $\mathrm{V}_0$ in the general equation of fluids (5), we obtain by the calculus of variations, the following result,

$$\iiint(\mathrm{X}\delta\xi + \mathrm{Y}\delta\eta + \mathrm{Z}\delta\zeta)dm = \iint p\delta\xi dydz + \iint p\delta\eta dxdz + \iint p\delta\zeta dxdy$$
$$- \iiint\left(\frac{dp}{dx}\delta\xi + \frac{dp}{dy}\delta\eta + \frac{dp}{dz}\delta\zeta\right)dx\,dy\,dz, \quad (8)$$

the triple integrals giving the equations of equilibrium, and the double integrals giving the conditions at the limits. If the density be expressed by $\epsilon$, so that $dm = \epsilon dxdydz$, we have

$$-\epsilon \mathrm{X} = \frac{dp}{dx}, \quad -\epsilon \mathrm{Y} = \frac{dp}{dy}, \quad -\epsilon \mathrm{Z} = \frac{dp}{dz}, \tag{9}$$

which are the well-known equations of hydrostatics. It appears from these equations that if the forces (X, Y, Z) be zero at all points of the fluid, the quantity $p$ must be constant; and *vice versa*, if $p$ be constant for all points of the fluid, the forces X, Y, Z must be zero; in such a case the function $\mathrm{V}_0$ will not give any indefinite equations of equilibrium, but only the particular condition at the limits expressed by the double integrals, which is the same as that found by Lagrange, and expresses that there must be a constant normal pressure at each point of the limiting surface, equal to $p$, in order that the fluid should remain in equilibrium.

The hydrodynamical equations corresponding to (9) are

$$\epsilon\frac{d^2\xi}{dt^2} = \epsilon \mathrm{X} + \frac{dp}{dx}, \quad \epsilon\frac{d^2\eta}{dt^2} = \epsilon \mathrm{Y} + \frac{dp}{dy}, \quad \epsilon\frac{d^2\zeta}{dt^2} = \epsilon \mathrm{Z} + \frac{dp}{dz}, \tag{10}$$

which are the equations commonly used in hydrodynamics, and, together with the equations (9), are usually deduced from the principle of equality of pressure, of which principle they are merely the mathematical expression: but the fact that this principle is often not true in the case of fluids in motion, would lead us to suppose that though the equations may do well enough as a first approximation, yet that in hydrodynamics they are *insufficient;* in fact we have, in general,

$$\mathrm{V} = \mathrm{V}_0 + \mathrm{V}_1 + \mathrm{V}_2 + \&\mathrm{c}.$$

and the terms $\mathrm{V}_0$, $\mathrm{V}_1$, &c. give rise, in the equations of equilibrium and motion, to differential coefficients of the first, second, &c., order; and in case the supposition $\mathrm{V} = \mathrm{V}_0$ becomes insufficient, we must then make $\mathrm{V} = \mathrm{V}_0 + \mathrm{V}_1$, and add to the differential equations of equilibrium and motion, arising from $\mathrm{V}_0$, the additional terms produced by $\mathrm{V}_1$. I shall return to this subject in a subsequent part of this paper, when I have determined the form of the terms produced by $\mathrm{V}_1$.

It appears from (4) that

$$\mathrm{V}_1 = \int_0^\infty\int_0^\pi\int_0^{2\pi} \mathrm{F}_1(\rho')^2.\rho^2\sin\theta\, d\rho\, d\theta\, d\phi,$$

from which the following value for $v_1$ may be deduced, by substituting for $(\rho')$ its value taken from (6),

$$2v_1 = A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2 + LU^2 + MV^2 + NW^2 \tag{11}$$

$$+2\left(L\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz} + M\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz} + N\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right) + 2(\alpha_1 VW + \beta_2 UW + \gamma_3 UV)$$

$$+2\left\{U\left(\alpha_1\frac{d\xi}{dx}+\beta_1\frac{d\eta}{dy}+\gamma_1\frac{d\zeta}{dz}\right)+V\left(\alpha_2\frac{d\xi}{dx}+\beta_2\frac{d\eta}{dy}+\gamma_2\frac{d\zeta}{dz}\right)+W\left(\alpha_3\frac{d\xi}{dx}+\beta_3\frac{d\eta}{dy}+\gamma_3\frac{d\zeta}{dz}\right)\right\}$$

and if we make $d\omega = \rho^4 \sin\theta\, d\rho\, d\theta\, d\phi$, we shall have the following values for the coefficients of the function $v_1$:

$A = 2\iiint F_1 . \cos^4\alpha\, d\omega$, $B = 2\iiint F_1 . \cos^4\beta\, d\omega$, $C = 2\iiint F_1 . \cos^4\gamma\, d\omega$,

$L = 2\iiint F_1 . \cos^2\beta\cos^2\gamma\, d\omega$, $M = 2\iiint F_1 . \cos^2\alpha\cos^2\gamma\, d\omega$, $N = 2\iiint F_1 . \cos^2\alpha\cos^2\beta\, d\omega$,

$\alpha_1 = 2\iiint F_1 . \cos^2\alpha\cos\beta\cos\gamma\, d\omega$, $\beta_1 = 2\iiint F_1 . \cos^3\beta\cos\gamma\, d\omega$, $\gamma_1 = 2\iiint F_1 . \cos^3\gamma\cos\beta\, d\omega$,

$\alpha_2 = 2\iiint F_1 . \cos^3\alpha\cos\gamma\, d\omega$, $\beta_2 = 2\iiint F_1 . \cos^2\beta\cos\alpha\cos\gamma\, d\omega$, $\gamma_2 = 2\iiint F_1 . \cos^3\gamma\cos\alpha\, d\omega$,

$\alpha_3 = 2\iiint F_1 . \cos^3\alpha\cos\beta\, d\omega$, $\beta_3 = 2\iiint F_1 . \cos^3\beta\cos\alpha\, d\omega$, $\gamma_3 = 2\iiint F_1 . \cos^2\gamma\cos\alpha\cos\beta\, d\omega$.

It appears, therefore, that the form of the function $v$ for solids (in which we have $v = v_1$), is that of a *homogeneous function of the second order* of the six quantities, $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$, $U$, $V$, $W$. The most general form of such a function would contain twenty-one constants, but the function $v$, which has been just determined, contains only fifteen constants, although it has the full number of terms. The general equation of equilibrium and motion of solid bodies is therefore

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iiint \delta v_1\, dx\, dy\, dz, \tag{12}$$

in which the expression (11) for $v_1$ must be substituted, and the equation treated in a manner analogous to what has been already done for equation (5), this will produce the general equations of equilibrium and motion, and also the conditions at the limits expressed by means of double integrals.

It is important to observe that the function $v_1$, which has been just found for solids, is quite different from the function $v$, used by Professor M'Cullagh in his Mechanical Theory of Light; in that theory the function $v$, which defines the medium whose vibrations produce the sensation of light, is a homogeneous function of the second order of the three quantities,

$$\frac{d\eta}{dz}-\frac{d\zeta}{dy}, \quad \frac{d\zeta}{dx}-\frac{d\xi}{dz}, \quad \frac{d\xi}{dy}-\frac{d\eta}{dx},$$

this form of the function being deduced from the equation

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0,$$

which expresses that the medium refuses to change its density.

It follows from this difference of form of the two functions, that in light the molecules of the medium cannot act on each other in the direction of the line joining them, for if they did, the function corresponding to the medium should contain terms depending on the quantities

$$\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}, \quad \text{U, V, W}.$$

Previously to investigating the differential equations of equilibrium and motion, produced by the function $V_1$, it is important to examine whether the function $V_1$ itself is capable of any simplification in its form by means of a change in the direction of the axes of coordinates, to which it is referred; as, if it be possible to refer it to particular axes, for which it becomes simplified, this will also simplify the equations of equilibrium and motion depending on it, and will prove the existence of three rectangular axes in the body at each point, which will have an intimate connexion with the molecular structure of the body at every point. The equations for transforming $\xi$, $\eta$, $\zeta$ are

$$\begin{aligned}\xi &= a\xi' + b\eta' + c\zeta',\\ \eta &= a'\xi' + b'\eta' + c'\zeta',\\ \zeta &= a''\xi' + b''\eta' + c''\zeta',\end{aligned}$$

the six quantities, $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$, U, V, W, must be transformed by the aid of these three equations, and the transformed values substituted in the function $V_1$; adopting, for brevity, the following expressions,

$$\begin{aligned}p &= 2a'a'', & q &= 2b'b'', & r &= 2c'c'',\\ p' &= 2aa'', & q' &= 2bb'', & r' &= 2cc'',\\ p'' &= 2aa', & q'' &= 2bb', & r'' &= 2cc',\end{aligned}$$

$$\begin{aligned} l\ &= b'c'' + b''c', & m\ &= c'a'' + c''a', & n\ &= a'b'' + a''b', \\ l'\ &= b''c\ + bc'', & m'\ &= c''a\ + ca'', & n'\ &= a''b + ab'', \\ l'' &= bc'\ + b'c, & m'' &= ca'\ + c'a, & n'' &= ab'\ + a'b, \end{aligned}$$

and, effecting the transformations, we shall obtain the following values for the six quantities, $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$, $\mathrm{u}$, $\mathrm{v}$, $\mathrm{w}$, expressed as linear functions of the six transformed quantities, $\frac{d\xi'}{dx'}, \frac{d\eta'}{dy'}, \frac{d\zeta'}{dz'}$, $\mathrm{u}'$, $\mathrm{v}'$, $\mathrm{w}'$.

$$\left.\begin{aligned} \frac{d\xi}{dx} &= a^2\frac{d\xi'}{dx'} + b^2\frac{d\eta'}{dy'} + c^2\frac{d\zeta'}{dz'} + bc\,.\mathrm{u}' + ac.\mathrm{v}' + ab\mathrm{w}', \\ \frac{d\eta}{dy} &= a'^2\frac{d\xi'}{dx'} + b'^2\frac{d\eta'}{dy'} + c'^2\frac{d\zeta'}{dz'} + b'c'.\mathrm{u}' + a'c'.\mathrm{v}' + a'b'\mathrm{w}', \\ \frac{d\zeta}{dz} &= a''^2\frac{d\xi'}{dx'} + b''^2\frac{d\eta'}{dy'} + c''^2\frac{d\zeta'}{dz'} + b''c''.\mathrm{u}' + a''c''.\mathrm{v}' + a''b''\mathrm{w}'. \\ \mathrm{u} &= p\frac{d\xi'}{dx'} + q\frac{d\eta'}{dy'} + r\frac{d\zeta'}{dz'} + l\mathrm{u}' + m\mathrm{v}' + n\mathrm{w}', \\ \mathrm{v} &= p'\frac{d\xi'}{dx'} + q'\frac{d\eta'}{dy'} + r'\frac{d\zeta'}{dz'} + l'\mathrm{u}' + m'\mathrm{v}' + n'\mathrm{w}', \\ \mathrm{w} &= p''\frac{d\xi'}{dx'} + q''\frac{d\eta'}{dy'} + r''\frac{d\zeta'}{dz'} + l''\mathrm{u}' + m''\mathrm{v}' + n''\mathrm{w}'. \end{aligned}\right\} \quad (13)$$

The substitution of these values in the function $\mathrm{v}_1$ will transform it into another form, which still retains the twenty-one terms, though now expressed in terms of the six new variables; the coefficients of which will be known functions of the nine angles which determine the position of the new axes of coordinates with respect to the old axes. Our object is to discover relations among these coefficients which will enable us, by satisfying them with *real* values of the nine cosines ($a$, $b$, $c$, $a'$, $b'$, $c'$, $a''$, $b''$, $c''$), to reduce the function $\mathrm{v}_1$ to a more simple form, and consequently to diminish the number of constants, which, in general, is fifteen.

Let ($\aleph_1, \beth_1, \gimel_1, \aleph_2, \beth_2, \gimel_2, \aleph_3, \beth_3, \gimel_3$) denote the transformed values of ($\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$); the values of these coefficients, when developed, are,

$$\begin{aligned}
\aleph_1 = {} & (\mathrm{A}a^2bc + \mathrm{B}a'^2b'c' + \mathrm{C}a''^2b''c'') + 3(\mathrm{L}la'a'' + \mathrm{M}l'aa'' + \mathrm{N}l''aa') \\
& + (\alpha_1(p'l'' + p''l') + \beta_2(p''l + pl'') + \gamma_3(pl' + p'l)) - (\mathrm{L}bc + \mathrm{M}b'c' + \mathrm{N}b''c'') \\
& + \alpha_1(la^2 + pbc) + \beta_1(la'^2 + pb'c') + \gamma_1(la''^2 + pb''c'') \\
& + \alpha_2(l'a^2 + p'bc) + \beta_2(l'a'^2 + p'b'c') + \gamma_2(l'a''^2 + p'b''c'') \\
& + \alpha_3(l''a^2 + p''bc) + \beta_3(l''a'^2 + p''b'c') + \gamma_3(l''a''^2 + p''b''c'')
\end{aligned}$$

$$\begin{aligned}
\beth_1 = {} & (\mathrm{A}b^3c + \mathrm{B}b'^3c' + \mathrm{C}b''^3c'') + 3(\mathrm{L}lb'b'' + \mathrm{M}l'bb'' + \mathrm{N}l''bb') \\
& + 3\alpha_1(lb^2 + qbc) + \beta_1(lb'^2 + qb'c') + \gamma_1(lb''^2 + qb''c'') \\
& + \alpha_2(l'b^2 + q'bc) + 3\beta_2(l'b'^2 + q'b'c') + \gamma_2(l'b''^2 + q'b''c'') \\
& + \alpha_3(l''b^2 + q''bc) + \beta_3(l''b'^2 + q''b'c') + 3\gamma_3(l''b''^2 + q''b''c'')
\end{aligned}$$

$$\begin{aligned}
\gimel_1 = {} & (\mathrm{A}c^3b + \mathrm{B}c'^3b' + \mathrm{C}c''^3b'') + 3(\mathrm{L}lc'c'' + \mathrm{M}l'cc'' + \mathrm{N}l''cc') \\
& + 3\alpha_1(lc^2 + rbc) + \beta_1(lc'^2 + rb'c') + \gamma_1(lc''^2 + rb''c'') \\
& + \alpha_2(l'c^2 + r'bc) + 3\beta_2(l'c'^2 + r'b'c') + \gamma_2(l'c''^2 + r'b''c'') \\
& + \alpha_3(l''c^2 + r''bc) + \beta_3(l''c'^2 + r''b'c') + 3\gamma_3(l''c''^2 + r''b''c'')
\end{aligned}$$

$$\begin{aligned}
\aleph_2 = {} & (\mathrm{A}a^3c + \mathrm{B}a'^3c' + \mathrm{C}a''^3c'') + 3(\mathrm{L}ma'a'' + \mathrm{M}m'aa'' + \mathrm{N}m''aa') \\
& + 3\alpha_1(ma^2 + pac) + \beta_1(ma'^2 + pa'c') + \gamma_1(ma''^2 + pa''c'') \\
& + \alpha_2(m'a^2 + p'ac) + 3\beta_2(m'a'^2 + p'a'c') + \gamma_2(m'a''^2 + p'a''c'') \\
& + \alpha_3(m''a^2 + p''ac) + \beta_3(m''a'^2 + p''a'c') + 3\gamma_3(m''a''^2 + p''a''c'')
\end{aligned} \tag{14}$$

$$\begin{aligned}
\beth_2 = {} & (\mathrm{A}b^2ac + \mathrm{B}b'^2a'c' + \mathrm{C}b''^2a''c'') + 3(\mathrm{L}mb'b'' + \mathrm{M}m'bb'' + \mathrm{N}m''bb') \\
& + (\alpha_1(q'm'' + q''m') + \beta_2(q''m + qm'') + \gamma_3(qm' + q'm)) - (\mathrm{L}ac + \mathrm{M}a'c' + \mathrm{N}a''c'') \\
& + \alpha_1(mb^2 + qac) + \beta_1(mb'^2 + qa'c') + \gamma_1(mb''^2 + qa''c'') \\
& + \alpha_2(m'b^2 + q'ac) + \beta_2(m'b'^2 + q'a'c') + \gamma_2(m'b''^2 + q'a''c'') \\
& + \alpha_3(m''b^2 + q''ac) + \beta_3(m''b'^2 + q''a'c') + \gamma_3(m''b''^2 + q''a''c'')
\end{aligned}$$

$$\begin{aligned}
\gimel_2 = {} & (\mathrm{A}c^3a + \mathrm{B}c'^3a' + \mathrm{C}c''^3a'') + 3(\mathrm{L}mc'c'' + \mathrm{M}m'cc'' + \mathrm{N}m''cc') \\
& + 3\alpha_1(mc^2 + rac) + \beta_1(mc'^2 + ra'c') + \gamma_1(mc''^2 + ra''c'') \\
& + \alpha_2(m'c^2 + r'ac) + 3\beta_2(m'c'^2 + r'a'c') + \gamma_2(m'c''^2 + r'a''c'') \\
& + \alpha_3(m''c^2 + r''ac) + \beta_3(m''c'^2 + r''a'c') + 3\gamma_3(m''c''^2 + r''a''c'')
\end{aligned}$$

$$\begin{aligned}\aleph_3 = {} & (\mathrm{A}a^3b + \mathrm{B}a'^3b' + \mathrm{C}a''^3b'') + 3(\mathrm{L}na'a'' + \mathrm{M}n'aa'' + \mathrm{N}n''aa') \\ & + 3\alpha_1(na^2 + pab) + \beta_1(na'^2 + pa'b') + \gamma_1(na''^2 + pa''b'') \\ & + \alpha_2(n'a^2 + p'ab) + 3\beta_2(n'a'^2 + p'a'b') + \gamma_2(n'a''^2 + p'a''b'') \\ & + \alpha_3(n''a^2 + p''ab) + \beta_3(n''a'^2 + p''a'b') + 3\gamma_3(n''a''^2 + p''a''b'')\end{aligned}$$

$$\begin{aligned}\beth_3 = {} & (\mathrm{A}b^3a + \mathrm{B}b'^3a' + \mathrm{C}b''^3a'') + 3(\mathrm{L}nb'b'' + \mathrm{M}n'bb'' + \mathrm{N}n''bb') \\ & + 3\alpha_1(nb^2 + qab) + \beta_1(nb'^2 + qa'b') + \gamma_1(nb''^2 + qa''b'') \\ & + \alpha_2(n'b^2 + q'ab) + 3\beta_2(n'b'^2 + q'a'b') + \gamma_2(n'b''^2 + q'a''b'') \\ & + \alpha_3(n''b^2 + q''ab) + \beta_3(n''b'^2 + q''a'b') + 3\gamma_3(n''b''^2 + q''a''b'')\end{aligned}$$

$$\begin{aligned}\gimel_3 = {} & (\mathrm{A}c^2ab + \mathrm{B}c'^2a'b' + \mathrm{C}c''^2a''b'') + 3(\mathrm{L}nc'c'' + \mathrm{M}n'cc'' + \mathrm{N}n''cc') \\ & + (\alpha_1(r'n'' + r''n') + \beta_2(r''n + rn'') + \gamma_3(rn' + r'n)) - (\mathrm{L}ab + \mathrm{M}a'b' + \mathrm{N}a''b'') \\ & + \alpha_1(nc^2 + rab) + \beta_1(nc'^2 + ra'b') + \gamma_1(nc''^2 + ra''b'') \\ & + \alpha_2(n'c^2 + r'ab) + \beta_2(n'c'^2 + r'a'b') + \gamma_2(n'c''^2 + r'a''b'') \\ & + \alpha_3(n''c^2 + r''ab) + \beta_3(n''c'^2 + r''a'b') + \gamma_3(n''c''^2 + r''a''b'')\end{aligned}$$

In obtaining the foregoing values of the coefficients, use has been made of the following relations among the nine angles:

$$\begin{aligned} a &= b'c'' - b''c', & a' &= b''c - bc'', & a'' &= bc' - b'c, \\ b &= c'a'' - c''a', & b' &= c''a - ca'', & b'' &= ca' - c'a, \\ c &= a'b'' - a''b', & c' &= a''b - ab'', & c'' &= ab' - a'b, \end{aligned}$$

which may be easily seen to be identically true.

It can be proved from the values of the nine coefficients which have been given, that there exists a *real* system of rectangular axes at each point of the body, for which the following equations are true,

$$\aleph_1 + \beth_1 + \gimel_1 = 0, \quad \aleph_2 + \beth_2 + \gimel_2 = 0, \quad \aleph_3 + \beth_3 + \gimel_3 = 0, \qquad (15)$$

for, we obtain from (14), since, $p + q + r = 0$, $p' + q' + r' = 0$, &c. &c.

$$\begin{aligned}\aleph_1 + \beth_1 + \gimel_1 = {} & (\mathrm{A} - \mathrm{L})bc + (\mathrm{B} - \mathrm{M})b'c' + (\mathrm{C} - \mathrm{N})b''c'' \\ & + (\alpha_1 + \beta_1 + \gamma_1)l + (\alpha_2 + \beta_2 + \gamma_2)l' + (\alpha_3 + \beta_3 + \gamma_3)l'', \\ \aleph_2 + \beth_2 + \gimel_2 = {} & (\mathrm{A} - \mathrm{L})ac + (\mathrm{B} - \mathrm{M})a'c' + (\mathrm{C} - \mathrm{N})a''c'' \\ & + (\alpha_1 + \beta_1 + \gamma_1)m + (\alpha_2 + \beta_2 + \gamma_2)m' + (\alpha_3 + \beta_3 + \gamma_3)m'', \\ \aleph_3 + \beth_3 + \gimel_3 = {} & (\mathrm{A} - \mathrm{L})ab + (\mathrm{B} - \mathrm{M})a'b' + (\mathrm{C} - \mathrm{N})a''b'' \\ & + (\alpha_1 + \beta_1 + \gamma_1)n + (\alpha_2 + \beta_2 + \gamma_2)n' + (\alpha_3 + \beta_3 + \gamma_3)n''.\end{aligned}$$

Hence, equations (15) are equivalent to

$$\begin{aligned}&(\text{A}-\text{L})bc+(\text{B}-\text{M})b'c'+(\text{C}-\text{N})b''c''\\&\quad+(\alpha_1+\beta_1+\gamma_1)l+(\alpha_2+\beta_2+\gamma_2)l'+(\alpha_3+\beta_3+\gamma_3)l''=0,\\&(\text{A}-\text{L})ac+(\text{B}-\text{M})a'c'+(\text{C}-\text{N})a''c''\\&\quad+(\alpha_1+\beta_1+\gamma_1)m+(\alpha_2+\beta_2+\gamma_2)m'+(\alpha_3+\beta_3+\gamma_3)m''=0,\quad(16)\\&(\text{A}-\text{L})ab+(\text{B}-\text{M})a'b'+(\text{C}-\text{N})a''b''\\&\quad+(\alpha_1+\beta_1+\gamma_1)n+(\alpha_2+\beta_2+\gamma_2)n'+(\alpha_3+\beta_3+\gamma_3)n''=0.\end{aligned}$$

It is possible to find a unique system of rectangular coordinates for which these equations shall be true; for assume the ellipsoid, whose equation is

$$\begin{aligned}&(\text{A}-\text{L})x^2+(\text{B}-\text{M})y^2+(\text{C}-\text{N})z^2\\&\quad+2(\alpha_1+\beta_1+\gamma_1)yz+2(\alpha_2+\beta_2+\gamma_2)xz+2(\alpha_3+\beta_3+\gamma_3)xy=1,\end{aligned}$$

and transform it by the formulæ,

$$\begin{aligned}x&=ax'+by'+cz',\\y&=a'x'+b'y'+c'z',\\z&=a''x'+b''y'+c''z',\end{aligned}$$

the coefficients of the rectangles will become the expressions (16), and, consequently, for axes of coordinates coinciding in direction with the axes of this ellipsoid, the equations (15), (16) will be satisfied.

The function $\text{v}_1$, referred to these axes of coordinates, will have the number of its constants diminished by three, and consequently the total number will ultimately be twelve; it does not appear possible to banish a greater number by any transformation with rectangular coordinates, as in this transformation we have only three independent variables; with oblique coordinates, however, we should have six, and it is quite possible that there should exist in solid bodies an oblique system of axes, which would simplify still further the form of the function $\text{v}_1$. Such a system of axes, as well as the rectangular system just determined, would have an intimate connexion with the molecular structure of the body; though it is not easy to see exactly what that connexion would denote.

It may be useful to notice here a remarkable relation which exists between the function $\text{v}_1$ and a certain surface of the fourth order, whose properties have a great similarity to those of the function; and the connexion between them is such that a knowledge of any properties of the one must throw a corresponding light on the other. This surface is,

$$\begin{aligned}&(\mathrm{A}x^4 + \mathrm{B}y^4 + \mathrm{C}z^4) + 6(\mathrm{L}y^2z^2 + \mathrm{M}x^2z^2 + \mathrm{N}x^2y^2) \qquad (17)\\ &+4yz(3\alpha_1x^2+\beta_1y^2+\gamma_1z^2)+4xz(\alpha_2x^2+3\beta_2y^2+\gamma_2z^2)+4xy(\alpha_3x^2+\beta_3y^2+3\gamma_3z^2)=1\end{aligned}$$

The relation existing between this surface and the function $\mathrm{V}_1$ is the following: that, if this surface be referred to a new system of rectangular coordinates, the new coefficients will be expressed by the same functions of the nine angles as the coefficients of the function $\mathrm{V}_1$, when similarly changed by a transformation of coordinates; so that the relations (15) may be established among the coefficients of the surface (17), and *vice versâ*, if we knew any geometrical properties which would reduce the surface, there would exist corresponding properties of the function $\mathrm{V}_1$, which would throw light on the nature of the medium represented by $\mathrm{V}_1$. The relation which exists between the function $\mathrm{V}_1$ and the surface (17) may be proved as follows; the surface may be put into the following form:

$$\begin{aligned}&[\mathrm{A}(x^2)^2 + \mathrm{B}(y^2)^2 + \mathrm{C}(z^2)^2] + [\mathrm{L}(2yz)^2 + \mathrm{M}(2xz)^2 + \mathrm{N}(2xy)^2]\\ &+ 2[\mathrm{L}(y^2)(z^2) + \mathrm{M}(x^2)(z^2) + \mathrm{N}(x^2)(y^2)] + 2[\alpha_1(2xy)(2xz) + \beta_2(2xy)(2yz)\\ &\qquad + \gamma_3(2xz)(2yz)]\\ &+ 2\left\{\begin{array}{l}(2yz)(a_1(x^2) + \beta_1(y^2) + \gamma_1(z^2))\\ (2xz)(a_2(x^2) + \beta_2(y^2) + \gamma_2(z^2))\\ (2xy)(a_3(x^2) + \beta_3(y^2) + \gamma_3(z^2))\end{array}\right\} = 1,\end{aligned}$$

which is the same function of the six quantities, $x^2$, $y^2$, $z^2$, $2yz$, $2xz$, $2xy$, that $\mathrm{V}_1$ is of $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$, U, V, W. Also, it may be easily seen that

$$\begin{aligned}x^2 &= a^2x'^2 + b^2y'^2 + c^2z'^2 + bc.(2y'z') + ac(2x'z') + ab(2x'y'),\\ y^2 &= a'^2x'^2 + b'^2y'^2 + c'^2z'^2 + b'c'(2y'z') + a'c'(2x'z') + a'b'(2x'y'),\\ z^2 &= a''^2x'^2 + b''^2y'^2 + c''^2z'^2 + b''c''(2y'z')+a''c''(2x'z')+a''b''(2x'y'), \quad (18)\\ 2yz &= px'^2 + qy'^2 + rz'^2 + l(2y'z') + m(2x'z') + n(2x'y'),\\ 2xz &= p'x'^2 + q'y'^2 + r'z'^2 + l'(2y'z') + m'(2x'z') + n'(2x'y'),\\ 2xy &= p''x'^2 + q''y'^2 + r''z'^2 + l''(2y'z') + m''(2x'z') + n''(2x'y').\end{aligned}$$

These equations of transformation, having the same coefficients as the equations (13), prove, that the transformation of the surface (17) will produce the same coefficients as the transformation of the function $\mathrm{V}_1$.

I shall now proceed to determine the differential equations of equilibrium and motion arising from (12), which is the general equation applicable to all solid bodies, and common to them and fluids.

Substituting for $\mathrm{v}_1$ its value (11), we obtain, by the calculus of variations,

$$
\begin{aligned}
\iiint \delta \mathrm{V}_1 dxdydz = {} & \iint\left(\mathrm{A}\frac{d\xi}{dx} + \mathrm{N}\frac{d\eta}{dy} + \mathrm{M}\frac{d\zeta}{dz} + a_1\mathrm{U} + a_2\mathrm{V} + a_3\mathrm{W}\right)\delta\xi dydz \\
& + \iint\left(a_3\frac{d\xi}{dx} + \beta_3\frac{d\eta}{dy} + \gamma_3\frac{d\zeta}{dz} + \beta_2\mathrm{U} + a_1\mathrm{V} + \mathrm{NW}\right)\delta\eta dydz \\
& + \iint\left(a_2\frac{d\xi}{dx} + \beta_2\frac{d\eta}{dy} + \gamma_2\frac{d\zeta}{dz} + \gamma_3\mathrm{U} + \mathrm{MV} + a_1\mathrm{W}\right)\delta\zeta dydz \\
& + \iint\left(\mathrm{B}\frac{d\eta}{dy} + \mathrm{L}\frac{d\zeta}{dz} + \mathrm{N}\frac{d\xi}{dx} + \beta_1\mathrm{U} + \beta_2\mathrm{V} + \beta_3\mathrm{W}\right)\delta\eta dxdz \\
& + \iint\left(a_1\frac{d\xi}{dx} + \beta_1\frac{d\eta}{dy} + \gamma_1\frac{d\zeta}{dz} + \mathrm{LU} + \gamma_3\mathrm{V} + \beta_2\mathrm{W}\right)\delta\zeta dxdz \\
& + \iint\left(a_3\frac{d\xi}{dx} + \beta_3\frac{d\eta}{dy} + \gamma_3\frac{d\zeta}{dz} + \beta_2\mathrm{U} + \alpha_1\mathrm{V} + \mathrm{NW}\right)\delta\xi dxdz \\
& + \iint\left(\mathrm{C}\frac{d\zeta}{dz} + \mathrm{M}\frac{d\xi}{dx} + \mathrm{L}\frac{d\eta}{dy} + \gamma_1\mathrm{U} + \gamma_2\mathrm{V} + \gamma_3\mathrm{W}\right)\delta\zeta dxdy \\
& + \iint\left(a_2\frac{d\xi}{dx} + \beta_2\frac{d\eta}{dy} + \gamma_2\frac{d\zeta}{dz} + \gamma_3\mathrm{U} + \mathrm{MV} + a_1\mathrm{W}\right)\delta\xi dxdy \\
& + \iint\left(a_1\frac{d\xi}{dx} + \beta_1\frac{d\eta}{dy} + \gamma_1\frac{d\zeta}{dz} + \mathrm{LU} + \gamma_3\mathrm{V} + \beta_2\mathrm{W}\right)\delta\eta dxdy \\
- \iiint & \left(\mathrm{A}\frac{d^2\xi}{dx^2} + \mathrm{N}\frac{d^2\xi}{dy^2} + \mathrm{M}\frac{d^2\xi}{dz^2} + 2\left(a_1\frac{d^2\xi}{dydz} + a_2\frac{d^2\xi}{dxdz} + a_3\frac{d^2\xi}{dxdy}\right)\right)\delta\xi dxdydz \\
- \iiint & \left(a_3\frac{d^2\eta}{dx^2} + \beta_3\frac{d^2\eta}{dy^2} + \gamma_3\frac{d^2\eta}{dz^2} + 2\left(a_1\frac{d^2\eta}{dxdz} + \mathrm{N}\frac{d^2\eta}{dxdy} + \beta_2\frac{d^2\eta}{dydz}\right)\right)\delta\xi dxdydz \\
- \iiint & \left(a_2\frac{d^2\zeta}{dx^2} + \beta_2\frac{d^2\zeta}{dy^2} + \gamma_2\frac{d^2\zeta}{dz^2} + 2\left(a_1\frac{d^2\zeta}{dxdy} + \gamma_3\frac{d^2\zeta}{dydz} + \mathrm{M}\frac{d^2\zeta}{dxdz}\right)\right)\delta\xi dxdydz
\end{aligned}
$$

$$-\iiint\left(\mathrm{B}\frac{d^2\eta}{dy^2}+\mathrm{L}\frac{d^2\eta}{dz^2}+\mathrm{N}\frac{d^2\eta}{dx^2}+2\left(\beta_2\frac{d^2\eta}{dxdz}+\beta_3\frac{d^2\eta}{dxdy}+\beta_1\frac{d^2\eta}{dydz}\right)\right)\delta\eta dxdydz$$

$$-\iiint\left(\beta_1\frac{d^2\zeta}{dy^2}+\gamma_1\frac{d^2\zeta}{dz^2}+\alpha_1\frac{d^2\zeta}{dx^2}+2\left(\beta_2\frac{d^2\zeta}{dxdy}+\mathrm{L}\frac{d^2\zeta}{dydz}+\gamma_3\frac{d^2\zeta}{dxdz}\right)\right)\delta\eta dxdydz$$

$$-\iiint\left(\beta_3\frac{d^2\xi}{dy^2}+\gamma_3\frac{d^2\xi}{dz^2}+\alpha_3\frac{d^2\xi}{dx^2}+2\left(\beta_2\frac{d^2\xi}{dydz}+\alpha_1\frac{d^2\xi}{dxdz}+\mathrm{N}\frac{d^2\xi}{dxdy}\right)\right)\delta\eta dxdydz$$

$$-\iiint\left(\mathrm{C}\frac{d^2\zeta}{dz^2}+\mathrm{M}\frac{d^2\zeta}{dx^2}+\mathrm{L}\frac{d^2\zeta}{dy^2}+2\left(\gamma_3\frac{d^2\zeta}{dxdy}+\gamma_1\frac{d^2\zeta}{dydz}+\gamma_2\frac{d^2\zeta}{dxdz}\right)\right)\delta\zeta dxdydz$$

$$-\iiint\left(\gamma_2\frac{d^2\xi}{dz^2}+\alpha_2\frac{d^2\xi}{dx^2}+\beta_2\frac{d^2\xi}{dy^2}+2\left(\gamma_3\frac{d^2\xi}{dydz}+\mathrm{M}\frac{d^2\xi}{dxdz}+\alpha_1\frac{d^2\xi}{dxdy}\right)\right)\delta\zeta dxdydz$$

$$-\iiint\left(\gamma_1\frac{d^2\eta}{dz^2}+\alpha_1\frac{d^2\eta}{dx^2}+\beta_1\frac{d^2\eta}{dy^2}+2\left(\gamma_3\frac{d^2\eta}{dxdz}+\beta_2\frac{d^2\eta}{dxdy}+\mathrm{L}\frac{d^2\eta}{dydz}\right)\right)\delta\zeta dxdydz$$

Hence, finally, equation (12) becomes

$$\iiint\epsilon(\mathrm{X}\delta\xi+\mathrm{Y}\delta\eta+\mathrm{Z}\delta\zeta)dxdydz=\Delta-\iiint(\mathrm{P}_1\delta\xi+\mathrm{Q}_1\delta\eta+\mathrm{R}_1\delta\zeta)dxdydz, \quad (19)$$

where $\epsilon$ denotes the density, $\Delta$ the sum of the nine double integrals, which belong exclusively to the limits, and $\mathrm{P}_1$, $\mathrm{Q}_1$, $\mathrm{R}_1$ the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$, in the triple integrals, which give the indefinite equations of equilibrium and motion. The equations of equilibrium and motion of solid bodies deduced from (19), are the following:

$$-\epsilon\mathrm{X}=\mathrm{P}_1,\quad -\epsilon\mathrm{Y}=\mathrm{Q}_1,\quad -\epsilon\mathrm{Z}=\mathrm{R}_1, \quad (20)$$

and

$$\epsilon\frac{d^2\xi}{dt^2}=\epsilon\mathrm{X}+\mathrm{P}_1,\quad \epsilon\frac{d^2\eta}{dt^2}=\epsilon\mathrm{Y}+\mathrm{Q}_1\quad \epsilon\frac{d^2\zeta}{dt^2}=\epsilon\mathrm{Z}+\mathrm{R}_1. \quad (21)$$

These are the most general equations of equilibrium and motion, and $\mathrm{P}_1$, $\mathrm{Q}_1$, $\mathrm{R}_1$ are expressed in their most general form, without making any supposition as to the arrangement of the molecules of the body.

In order to express the conditions at the limits produced by $\Delta$, it will be convenient to make use of the following transformation of the equation (19). Let

$$\aleph=\frac{d\mathrm{V}_1}{d\left(\frac{d\xi}{dx}\right)}=\mathrm{A}\frac{d\xi}{dx}+\mathrm{N}\frac{d\eta}{dy}+\mathrm{M}\frac{d\zeta}{dz}+\alpha_1\mathrm{U}+\alpha_2\mathrm{V}+\alpha_3\mathrm{W}$$

$$\beth = \frac{d\mathrm{v}_1}{d\left(\frac{d\eta}{dy}\right)} = \mathrm{B}\frac{d\eta}{dy} + \mathrm{L}\frac{d\zeta}{dz} + \mathrm{N}\frac{d\xi}{dx} + \beta_1\mathrm{U} + \beta_2\mathrm{V} + \beta_3\mathrm{W}$$

$$\gimel = \frac{d\mathrm{v}_1}{d\left(\frac{d\zeta}{dz}\right)} = \mathrm{C}\frac{d\zeta}{dz} + \mathrm{M}\frac{d\xi}{dx} + \mathrm{L}\frac{d\eta}{dy} + \gamma_1\mathrm{U} + \gamma_2\mathrm{V} + \gamma_3\mathrm{W}$$

$$\text{ל} = \frac{d\mathrm{v}_1}{d\mathrm{U}} = a_1\frac{d\xi}{dx} + \beta_1\frac{d\eta}{dy} + \gamma_1\frac{d\zeta}{dz} + \mathrm{LU} + \gamma_3\mathrm{V} + \beta_2\mathrm{W}$$

$$\text{מ} = \frac{d\mathrm{v}_1}{d\mathrm{V}} = a_2\frac{d\xi}{dx} + \beta_2\frac{d\eta}{dy} + \gamma_2\frac{d\zeta}{dz} + \gamma_3\mathrm{U} + \mathrm{MV} + a_1\mathrm{W}$$

$$\text{כ} = \frac{d\mathrm{v}_1}{d\mathrm{W}} = a_3\frac{d\xi}{dx} + \beta_3\frac{d\eta}{dy} + \gamma_3\frac{d\zeta}{dz} + \beta_2\mathrm{U} + a_1\mathrm{V} + \mathrm{NW}$$

It appears, from inspection of equation (19), that it may be thrown into the form,

$$\begin{aligned}\iiint \epsilon(\mathrm{X}\delta\xi + \mathrm{Y}\delta\eta + \mathrm{Z}\delta\zeta)dxdydz = & \iint(\aleph\delta\xi + \text{כ}\delta\eta + \text{מ}\delta\zeta)dydz \\ & + \iint(\beth\delta\eta + \text{ל}\delta\zeta + \text{כ}\delta\xi)dxdz \qquad (22) \\ & + \iint(\gimel\delta\zeta + \text{מ}\delta\xi + \text{ל}\delta\eta)dxdy\end{aligned}$$

$$-\iiint\left(\left(\frac{d\aleph}{dx} + \frac{d\text{כ}}{dy} + \frac{d\text{מ}}{dz}\right)\delta\xi + \left(\frac{d\beth}{dy} + \frac{d\text{ל}}{dz} + \frac{d\text{כ}}{dx}\right)\delta\eta + \left(\frac{d\gimel}{dz} + \frac{d\text{מ}}{dx} + \frac{d\text{ל}}{dy}\right)\delta\zeta\right)dxdydz;$$

and, consequently, the equations of equilibrium and motion become

$$\begin{aligned}\epsilon\mathrm{X} + \frac{d\aleph}{dx} + \frac{d\text{כ}}{dy} + \frac{d\text{מ}}{dz} &= 0 \\ \epsilon\mathrm{Y} + \frac{d\beth}{dy} + \frac{d\text{ל}}{dz} + \frac{d\text{כ}}{dx} &= 0 \qquad (23) \\ \epsilon\mathrm{Z} + \frac{d\gimel}{dz} + \frac{d\text{מ}}{dx} + \frac{d\text{ל}}{dy} &= 0\end{aligned}$$

and

$$\begin{aligned}\epsilon\frac{d^2\xi}{dt^2} &= \epsilon\mathrm{X} + \frac{d\aleph}{dx} + \frac{d\text{כ}}{dy} + \frac{d\text{מ}}{dz} \\ \epsilon\frac{d^2\eta}{dt^2} &= \epsilon\mathrm{Y} + \frac{d\beth}{dy} + \frac{d\text{ל}}{dz} + \frac{d\text{כ}}{dx} \qquad (24) \\ \epsilon\frac{d^2\zeta}{dt^2} &= \epsilon\mathrm{Z} + \frac{d\gimel}{dz} + \frac{d\text{מ}}{dx} + \frac{d\text{ל}}{dy}\end{aligned}$$

If $\omega$ represent the element of the bounding surface, and $(\lambda, \mu, \nu)$ the angles made by the normal with axes of coordinates, and $(\Xi, \Phi, \Psi)$ the *particular* forces acting at the limits of the solid; the equation of condition to be satisfied by the limits will be

$$\iint(\Xi\delta\xi + \Phi\delta\eta + \Psi\delta\zeta)\omega = \Delta;$$

but we have

$$\Delta = \iint(\aleph\cos\lambda + \text{כ}\cos\mu + \text{מ}\cos\nu)\omega\delta\xi + \iint(\beth\cos\mu + \text{ל}\cos\nu + \text{כ}\cos\lambda)\omega\delta\eta$$
$$+ \iint(\gimel\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu)\omega\delta\zeta$$

because

$$dydz = \omega\cos\lambda, \quad dxdz = \omega\cos\mu, \quad dxdy = \omega\cos\nu,$$

therefore

$$\begin{aligned} \Xi &= \aleph\cos\lambda + \text{כ}\cos\mu + \text{מ}\cos\nu \\ \Phi &= \beth\cos\mu + \text{ל}\cos\nu + \text{כ}\cos\lambda \\ \Psi &= \gimel\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu \end{aligned} \tag{25}$$

will be the equations of condition to be satisfied at the limiting surface. If the limits be fixed, then we shall have $\Delta = 0$, without any equation of condition; and if the surface be perfectly free, we shall have

$$\begin{aligned} \aleph\cos\lambda + \text{כ}\cos\mu + \text{מ}\cos\nu &= 0 \\ \beth\cos\mu + \text{ל}\cos\nu + \text{כ}\cos\lambda &= 0 \\ \gimel\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu &= 0 \end{aligned}$$

I shall now proceed to the laws of propagation and transmission of waves in an elastic crystalline solid, no external forces acting; and afterwards considèr some more particular cases of the theory, which will tend to throw light on the general method used in this Paper.

The differential equations of motion to be integrated are:

$$\epsilon\frac{d^2\xi}{dt^2} = A\frac{d^2\xi}{dx^2} + N\frac{d^2\xi}{dy^2} + M\frac{d^2\xi}{dz^2} + 2\left(a_1\frac{d^2\xi}{dydz} + a_2\frac{d^2\xi}{dxdz} + a_3\frac{d^2\xi}{dxdy}\right)$$
$$+ a_3\frac{d^2\eta}{dx^2} + \beta_3\frac{d^2\eta}{dy^2} + \gamma_3\frac{d^2\eta}{dz^2} + 2\left(a_1\frac{d^2\eta}{dxdz} + N\frac{d^2\eta}{dxdy} + \beta_2\frac{d^2\eta}{dydz}\right)$$
$$+ a_2\frac{d^2\zeta}{dx^2} + \beta_2\frac{d^2\zeta}{dy^2} + \gamma_2\frac{d^2\zeta}{dz^2} + 2\left(a_1\frac{d^2\zeta}{dxdy} + \gamma_3\frac{d^2\zeta}{dydz} + M\frac{d^2\zeta}{dxdz}\right)$$

$$\begin{aligned}
\epsilon\frac{d^2\eta}{dt^2} = {} & \mathrm{B}\frac{d^2\eta}{dy^2} + \mathrm{L}\frac{d^2\eta}{dz^2} + \mathrm{N}\frac{d^2\eta}{dx^2} + 2\left(\beta_2\frac{d^2\eta}{dxdz} + \beta_3\frac{d^2\eta}{dxdy} + \beta_1\frac{d^2\eta}{dydz}\right) \\
& + \beta_1\frac{d^2\zeta}{dy^2} + \gamma_1\frac{d^2\zeta}{dz^2} + \alpha_1\frac{d^2\zeta}{dx^2} + 2\left(\beta_2\frac{d^2\zeta}{dxdy} + \mathrm{L}\frac{d^2\zeta}{dydz} + \gamma_3\frac{d^2\zeta}{dxdz}\right) \\
& + \beta_3\frac{d^2\xi}{dy^2} + \gamma_3\frac{d^2\xi}{dz^2} + \alpha_3\frac{d^2\xi}{dx^2} + 2\left(\beta_2\frac{d^2\xi}{dydz} + \alpha_1\frac{d^2\xi}{dxdz} + \mathrm{N}\frac{d^2\xi}{dxdy}\right),
\end{aligned} \tag{26}$$

$$\begin{aligned}
\epsilon\frac{d^2\zeta}{dt^2} = {} & \mathrm{C}\frac{d^2\zeta}{dz^2} + \mathrm{M}\frac{d^2\zeta}{dx^2} + \mathrm{L}\frac{d^2\zeta}{dy^2} + 2\left(\gamma_3\frac{d^2\zeta}{dxdy} + \gamma_1\frac{d^2\zeta}{dydz} + \gamma_2\frac{d^2\zeta}{dxdz}\right) \\
& + \gamma_2\frac{d^2\xi}{dz^2} + \alpha_2\frac{d^2\xi}{dx^2} + \beta_2\frac{d^2\xi}{dy^2} + 2\left(\gamma_3\frac{d^2\xi}{dydz} + \mathrm{M}\frac{d^2\xi}{dxdz} + \alpha_1\frac{d^2\xi}{dxdy}\right) \\
& + \gamma_1\frac{d^2\eta}{dz^2} + \alpha_1\frac{d^2\eta}{dx^2} + \beta_1\frac{d^2\eta}{dy^2} + 2\left(\gamma_3\frac{d^2\eta}{dxdz} + \beta_2\frac{d^2\eta}{dxdy} + \mathrm{L}\frac{d^2\eta}{dydz}\right).
\end{aligned}$$

To these equations of motion we must add the equation of continuity,

$$\frac{d\epsilon'}{dt} + \epsilon'\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) = 0, \tag{27}$$

and we shall have four equations to determine the four quantities $(\xi, \eta, \zeta, \epsilon')$ as functions of $(x, y, z, t)$, $\epsilon$ being the original density, and $\epsilon'$ the variable density at any point. The general integral of these equations is hardly to be expected in the present state of the integral calculus; but there exists a particular integral, which represents the case of plane waves, and which leads to many important results. Let

$$\xi = \cos\alpha . f(\phi), \quad \eta = \cos\beta . f(\phi), \quad \zeta = \cos\gamma . f(\phi), \quad \phi = lx + my + nz - vt,$$

where $(\alpha, \beta, \gamma)$ denote the direction of vibration of molecules, $(l, m, n)$ the direction of normal to wave-plane, and $v$ the velocity of waves. The question to be determined is the following: given $(l, m, n)$, to determine, by substituting $(\xi, \eta, \zeta)$ in the equations of motion, whether it is possible to find real values of $(\alpha, \beta, \gamma, v)$, for which such a direction of wave-plane is possible.

Substituting the values of $(\xi, \eta, \zeta)$ in the equations (26), we obtain the following equations of condition:

$$\begin{aligned}
\epsilon v^2 . \cos\alpha &= \mathrm{P}'\cos\alpha + \mathrm{H}'\cos\beta + \mathrm{G}'\cos\gamma, \\
\epsilon v^2 . \cos\beta &= \mathrm{Q}'\cos\beta + \mathrm{F}'\cos\gamma + \mathrm{H}'\cos\alpha, \\
\epsilon v^2 . \cos\gamma &= \mathrm{R}'\cos\gamma + \mathrm{G}'\cos\alpha + \mathrm{F}'\cos\beta;
\end{aligned} \tag{28}$$

where

$$
\begin{aligned}
\mathrm{P}' &= \mathrm{A}l^2 + \mathrm{N}m^2 + \mathrm{M}n^2 + 2\alpha_1 mn + 2\alpha_2 ln + 2\alpha_3 lm,\\
\mathrm{Q}' &= \mathrm{B}m^2 + \mathrm{L}n^2 + \mathrm{N}l^2 + 2\beta_1 mn + 2\beta_2 ln + 2\beta_3 lm,\\
\mathrm{R}' &= \mathrm{C}n^2 + \mathrm{M}l^2 + \mathrm{L}m^2 + 2\gamma_1 mn + 2\gamma_2 ln + 2\gamma_3 lm,\\
\mathrm{F}' &= \alpha_1 l^2 + \beta_1 m^2 + \gamma_1 n^2 + 2\mathrm{L}mn + 2\gamma_3 ln + 2\beta_2 lm,\\
\mathrm{G}' &= \alpha_2 l^2 + \beta_2 m^2 + \gamma_2 n^2 + 2\gamma_3 mn + 2\mathrm{M}ln + 2\alpha_1 lm,\\
\mathrm{H}' &= \alpha_3 l^2 + \beta_3 m^2 + \gamma_3 n^2 + 2\beta_2 mn + 2\alpha_1 ln + 2\mathrm{N}lm.
\end{aligned}
\tag{29}
$$

Now, it is a known property of surfaces of the second order, that if the equation of a surface be

$$\mathrm{P}'x^2 + \mathrm{Q}'y^2 + \mathrm{R}'z^2 + 2\mathrm{F}'yz + 2\mathrm{G}'xz + 2\mathrm{H}'xy = 1, \tag{30}$$

that the equations (28) will determine the directions and magnitudes of the axes of the surface:* the directions being $(\alpha, \beta, \gamma)$, which have three sets of values; and also, if $(a, b, c)$ denote the lengths of the axes, they are determined by the equation,

$$\epsilon v^2 = \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2},$$

$\epsilon v^2$ having three values corresponding to the three sets of values of $(\alpha, \beta, \gamma)$.

Hence we may deduce a geometrical construction for the direction of vibration of molecules corresponding to a given direction of wave-plane.

Construct the six fixed ellipsoids:

$$
\begin{aligned}
\mathrm{P} &= \mathrm{A}x^2 + \mathrm{N}y^2 + \mathrm{M}z^2 + 2\alpha_1 yz + 2\alpha_2 xz + 2\alpha_3 xy = 1,\\
\mathrm{Q} &= \mathrm{B}y^2 + \mathrm{L}z^2 + \mathrm{N}x^2 + 2\beta_1 yz + 2\beta_2 xz + 2\beta_3 xy = 1,\\
\mathrm{R} &= \mathrm{C}z^2 + \mathrm{M}x^2 + \mathrm{L}y^2 + 2\gamma_1 yz + 2\gamma_2 yz + 2\gamma_3 xy = 1,\\
\mathrm{F} &= \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + 2\mathrm{L}yz + 2\gamma_3 xz + 2\beta_2 xy = 1,\\
\mathrm{G} &= \alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 + 2\gamma_3 yz + 2\mathrm{M}xz + 2\alpha_1 xy = 1,\\
\mathrm{H} &= \alpha_3 x^2 + \beta_3 y^2 + \gamma_3 z^2 + 2\beta_2 yz + 2\alpha_1 xz + 2\mathrm{N}xy = 1;
\end{aligned}
\tag{29}
$$

and from their common centre draw the normal to the wave-plane, this will pierce the surfaces in six points; let the corresponding radii vectores be $(\rho_{\prime}, \rho_{\prime\prime}, \rho_{\prime\prime\prime}, r_{\prime}, r_{\prime\prime}, r_{\prime\prime\prime})$, with these construct the ellipsoid,

$$\frac{x^2}{\rho_{\prime}^2} + \frac{y^2}{\rho_{\prime\prime}^2} + \frac{z^3}{\rho_{\prime\prime\prime}^2} + 2\left(\frac{yz}{r_{\prime}^2} + \frac{xz}{r_{\prime\prime}^2} + \frac{xy}{r_{\prime\prime\prime}^2}\right) = 1; \tag{30}$$

---

* Vid. Leroy, pp. 73, 156.

the directions of the axes of this surface are the three possible directions of molecular vibration for the given wave-plane; so that, for a given direction of wave-plane, *three* waves are possible, the corresponding vibrations of the molecules of the solid being at right angles to each other. The truth of this construction appears from the relations,

$$\mathrm{P}' = \frac{1}{\rho_{,}^{2}},\ \mathrm{Q}' = \frac{1}{\rho_{,,}^{2}},\ \mathrm{R}' = \frac{1}{\rho_{,,,}^{2}}.\ \mathrm{F}' = \frac{1}{r_{,}^{2}},\ \mathrm{G}' = \frac{1}{r_{,,}^{2}},\ \mathrm{H}' = \frac{1}{r_{,,,}^{2}},$$

which are easily proved. Also, since we may suppose all the parts of the vibrating body contained in a plane parallel to $lx + my + nz = 0$, to be of the same density during the progress of the moving plane which contains them, we shall have the three velocities of wave-propagation inversely proportional to the three axes of the ellipsoid (30).*

If wave normals were drawn from the common centre of the ellipsoids (29) in every possible direction, and the corresponding ellipsoids (30) constructed for each direction, and if on each normal three intercepts were measured, reciprocally proportional to the axes of the ellipsoid, the locus of the extremities of these intercepts would be the surface of wave-velocity, or locus of feet of perpendiculars from centre on tangent planes of wave-surface; and the surface formed by radii vectores, which are the reciprocals of the wave velocities, will be the surface of wave-slowness, or the reciprocal polar of the wave-surface, and a knowledge of its properties will serve all the purposes of the wave-surface itself; it is analogous to the index-surface in the theory of light. The surfaces of wave-velocity and wave-slowness may be determined by the following considerations:

The cubic equation whose roots are the squares of the reciprocals of the axes of the ellipsoid (30) is,

$$(\mathrm{P}'-s)\,(\mathrm{Q}'-s)\,(\mathrm{R}'-s) - \mathrm{F}'^{2}(\mathrm{P}'-s) - \mathrm{G}'^{2}(\mathrm{Q}'-s) - \mathrm{H}'^{2}(\mathrm{R}'-s) + 2\mathrm{F}'\mathrm{G}'\mathrm{H}' = 0.$$

If in this equation we substitute $(x^2 + y^2 + z^2)$ for $s$, and $\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right)$ for $(l, m, n)$, we shall have the equation of the surface, whose radii vectores measure the wave velocities; and if a radius vector be drawn from the centre, it will pierce this surface in three points, and the lengths of the three radii vectores measure the three velocities of wave-propagation possible for the given direction. This surface, however, being only the locus of feet of perpendiculars or tangent planes to wave-surface, is of very little use, but it enables us to determine the reciprocal polar of the wave-surface; for, if the radii vectores be made the reciprocals of their lengths, the direction remaining the same, we shall have for the locus of their extremities the surface of wave-slowness. Its equation is,

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* After having obtained this construction, I learned that M. Cauchy had given a similar one for a particular case; but he has not applied his results to their proper object, and has been misled in his attempt to account for the phenomena of light by his equations: his method, also, is quite different from that used in this Paper.

$$(P'\rho^2-1)\,(Q'\rho^2-1)\,(R'\rho^2-1)-F'^2\rho^4(P'\rho^2-1)-G'^2\rho^4(Q'\rho^2-1)-H'^2\rho^4(R'\rho^2-1)+2F'G'H'.\rho^6=0,$$

or,

$$(P-1)\,(Q-1)\,(R-1)-F^2(P-1)-G^2(Q-1)-H^2(R-1)+2FGH=0, \quad (31)$$

$(P, Q, R, F, G, H)$ being the quantities given in (29).

This surface is of the *sixth* degree, and has three sheets, corresponding to the three velocities of wave-propagation, and determines not merely the laws of propagation of plane waves in a solid, but also enables us to give a construction for the direction of waves refracted in passing from one solid into another.

With a point in the surface of separation (supposed plane) as centre, construct the two surfaces (31) for both solids; produce the normal to the incident wave to meet its own surface, and from the point in which it pierces it, let fall a perpendicular on the separating plane; this perpendicular will pierce the surface (31) of the second solid in three points; the lines drawn from these points to the centre are the normals of the three refracted waves, and their lengths are inversely proportional to the wave velocities.

The directions of the internal reflected waves may also be easily found by means of this surface; for we have only to produce the perpendicular backwards to meet the surface (31) of the first solid; and the three lines joining the centre with the three points of intersection will be the normals to the three plane waves reflected back into the first solid. These and all other constructions for the direction of reflected and refracted waves may be easily proved by the properties of the wave-surface and the reciprocal properties of the surface of wave-slowness, which, for such constructions as these, serves all the purposes of the wave-surface itself. The whole theory of wave-surfaces, in light, or in elastic solids, is only a development of Huygen's construction for uniaxal crystals.

It is important to observe that the directions of the *wave* and *ray*, being both given, (by the radius vector and perpendicular on tangent plane of the surface of wave-slowness), completely determine the direction of molecular vibration in any given case. If the direction of the *wave* alone be given, the problem is indeterminate; for three parallel tangent planes may be drawn to the wave surface, each tangent plane being accompanied by its own direction of molecular vibration, and the three directions of vibrations being at right angles to each other: but if the direction of the *ray* be also given, there will be nothing indeterminate, for this will show which of the three tangent planes we must select, and, consequently, which of the three directions of molecular vibration.

We now come to the conditions which must be satisfied at the limits; the facilities for obtaining which constitute the great advantage of the method of the *Mecanique Analytique*, as compared with any other method in Mechanics. The principal cases of limits which occur are the following: 1st. The limits may be completely fixed; 2nd. The body may have particular forces ($\Xi$, $\Phi$, $\Psi$) applied throughout the whole extent of the limiting surface; 3rd. The limits may be perfectly free; 4th. The limits may be the surfaces separating different kinds of bodies, so that the vibration passes from one into the other, changing the laws of its propagation at the limits, and requiring particular conditions to be satisfied at the separating surfaces.

1st. If the external surface of the body be completely fixed, so that its equation is independent of the time, then there are two cases to be considered: the molecules constituting the external surface may be absolutely fixed, so that no motion whatever is possible at the limits; or the external surface may be fixed, so that motion perpendicular to it is impossible; but there may be motion along the surface itself. In the first case we have no condition at the limits; for the condition to be satisfied at the limits being,

$$0 = \Delta = \iint\left(\left(\aleph\cos\lambda + \text{כ}\cos\mu + \text{מ}\cos\nu\right)\delta\xi + \left(\text{ב}\cos\mu + \text{ל}\cos\nu + \text{כ}\cos\lambda\right)\delta\eta \right.$$
$$\left. + \left(\text{ג}\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu\right)\delta\zeta\right)\omega \qquad (32)$$

is identically true, because

$$\delta\xi = 0,\ \ \delta\eta = 0,\ \ \delta\zeta = 0.$$

But in the second case (which occurs, for example, in hydrodynamics), the variations at the limiting surfaces are not zero, but only subjected to the restriction that they must take place along a given surface. Let the limiting surface be

$$f(x, y, z) = 0,$$

then the condition to be added to $\Delta = 0$, is

$$\frac{df}{dx}\delta\xi + \frac{df}{dy}\delta\eta + \frac{df}{dz}\delta\zeta = 0;$$

and, since $(\cos\lambda, \cos\mu, \cos\nu)$ are proportional to $\left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right)$, we shall obtain, finally, for the conditions at the limits,

$$\left(\text{א}\frac{df}{dx} + \text{נ}\frac{df}{dy} + \text{מ}\frac{df}{dz}\right)\delta\xi + \left(\text{ב}\frac{df}{dy} + \text{ל}\frac{df}{dz} + \text{נ}\frac{df}{dx}\right)\delta\eta + \left(\text{ג}\frac{df}{dz} + \text{מ}\frac{df}{dx} + \text{ל}\frac{df}{dy}\right)\delta\zeta = 0$$

$$\frac{df}{dx}\delta\xi + \frac{df}{dy}\delta\eta + \frac{df}{dz}\delta\zeta = 0, \tag{33}$$

which, by eliminating any one of the variations, will give ultimately the *two* conditions to be satisfied. It is evident, *a priori*, that in this case there must be two conditions at the limits, as the equation of the external surface is a function of two independent variables.

2nd. If particular forces $(\Xi, \Phi, \Psi)$ act at the external surface, then the equation of condition at the limits will be

$$\Delta = \iint(\Xi\delta\xi + \Phi\delta\eta + \Psi\delta\zeta)\,\omega,$$

or,

$$\begin{aligned} \Xi &= \text{א}\cos\lambda + \text{נ}\cos\mu + \text{מ}\cos\nu \\ \Phi &= \text{ב}\cos\mu + \text{ל}\cos\nu + \text{נ}\cos\lambda \\ \Psi &= \text{ג}\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu \end{aligned} \tag{34}$$

which are the same as equations (25).

3rd. If the limiting surface be perfectly free, then $(\delta\xi, \delta\eta, \delta\zeta)$ are independent of each other, and the general condition $\Delta = 0$ will give the *three* conditions,

$$\begin{aligned} \text{א}\cos\lambda + \text{נ}\cos\mu + \text{מ}\cos\nu &= 0 \\ \text{ב}\cos\mu + \text{ל}\cos\nu + \text{נ}\cos\lambda &= 0 \\ \text{ג}\cos\nu + \text{מ}\cos\lambda + \text{ל}\cos\mu &= 0 \end{aligned} \tag{35}$$

or, if $u = f(x, y, z, t)$ be the external surface,

$$\text{א}\frac{du}{dx}+\text{נ}\frac{du}{dy}+\text{מ}\frac{du}{dz}=0$$

$$\text{ב}\frac{du}{dy}+\text{ל}\frac{du}{dz}+\text{נ}\frac{du}{dx}=0$$

$$\text{ג}\frac{du}{dz}+\text{מ}\frac{du}{dx}+\text{ל}\frac{du}{dy}=0$$

4th. This case, in which we have two media in contact, is the most important and difficult of all the cases of conditions at limits. In order to discuss it, it is necessary to go back to the general equation of motion and equilibrium (12); in the case at present considered, the function $v_1$ must be conceived to be composed of two parts $(v_1', v_1'')$, $v_1'$ belonging to one body, and $v_1''$ to the other. Let $f(x,y,z)=0$ represent the surface passing through the molecules common to both media in their state of rest. There will then be three general equations of motion, belonging respectively to the molecules, for which

$$f(x,y,z)>0,\quad f(x,y,z)<0,\quad f(x,y,z)=0,$$

i. e. according as the molecules belong to the first medium, for which $v_1=v_1'$; or to the second, for which $v_1=v_1''$; or to both, for which case we have $v_1=v_1'+v_1''$. These three equations are

$$\iiint -\epsilon'\left(\frac{d^2\xi'}{dt^2}\delta\xi'+\frac{d^2\eta'}{dt^2}\delta\eta'+\frac{d^2\zeta'}{dt^2}\delta\zeta'\right)dxdydz=\iiint\delta v_1'dxdydz \tag{36}$$

$$\iiint -\epsilon''\left(\frac{d^2\xi''}{dt^2}\delta\xi''+\frac{d^2\eta''}{dt^2}\delta\eta''+\frac{d^2\zeta''}{dt^2}\delta\zeta''\right)dxdydz=\iiint\delta v_1''dxdydz \tag{37}$$

$$\begin{aligned}&\iiint -\epsilon'\left(\frac{d^2\xi'}{dt^2}\delta\xi'+\frac{d^2\eta'}{dt^2}\delta\eta'+\frac{d^2\zeta'}{dt^2}\delta\zeta'\right)dxdydz\\&+\iiint -\epsilon''\left(\frac{d^2\xi''}{dt^2}\delta\xi''+\frac{d^2\eta''}{dt^2}\delta\eta''+\frac{d^2\zeta''}{dt^2}\delta\zeta''\right)dxdydz\\&=\iiint\delta v_1'dxdydz+\iiint\delta v_1''dxdydz;\end{aligned} \tag{38}$$

the single and double marks denoting that the quantities belong to the first or second body respectively. As the laws of propagation of waves and the limiting conditions arising from fixed or free surfaces respectively have been already considered, we have at present nothing to do with equations (36) or (37), or with the triple integrals in (38), but only with the double integrals in equation (38), produced, as in (19), by integrating the right hand member by parts, according to the rules of the Calculus of Variations.

It is easy to see that the condition at the limits will be

$$\Delta = \Delta' - \Delta'' = 0,$$

$\Delta'$, $\Delta''$ arising from the two parts of (38), and having the form of $\Delta$ in equation (32). This condition, expressed at full length, is evidently equivalent to the equations

$$\left(\text{א}'\frac{du}{dx} + \text{נ}'\frac{du}{dy} + \text{מ}'\frac{du}{dz}\right)\delta\xi' - \left(\text{א}''\frac{du}{dx} + \text{נ}''\frac{du}{dy} + \text{מ}''\frac{du}{dz}\right)\delta\xi'' = 0$$

$$\left(\text{ב}'\frac{du}{dy} + \text{ל}'\frac{du}{dz} + \text{נ}'\frac{du}{dx}\right)\delta\eta' - \left(\text{ב}''\frac{du}{dy} + \text{ל}''\frac{du}{dz} + \text{נ}''\frac{du}{dx}\right)\delta\eta'' = 0$$

$$\left(\text{ג}'\frac{du}{dz} + \text{מ}'\frac{du}{dx} + \text{ל}'\frac{du}{dy}\right)\delta\zeta' - \left(\text{ג}''\frac{du}{dz} + \text{מ}''\frac{du}{dx} + \text{ל}''\frac{du}{dy}\right)\delta\zeta'' = 0$$

But as the motion of the molecules which constitute the surface $f(x, y, z) = 0$ is the same, whichever medium they are supposed to belong to, we shall have evidently $\xi' = \xi''$, $\eta' = \eta''$, $\zeta' = \zeta''$; and, expressing these in the equations just found, and equating to zero the coefficients of the arbitrary variations, we shall obtain finally the *three* following conditions, to be satisfied at the surface which separates two contiguous bodies.

$$(\text{א}' - \text{א}'')\frac{du}{dx} + (\text{נ}' - \text{נ}'')\frac{du}{dy} + (\text{מ}' - \text{מ}'')\frac{du}{dz} = 0$$

$$(\text{ב}' - \text{ב}'')\frac{du}{dy} \quad (\text{ל}' - \text{ל}'')\frac{du}{dz} + (\text{נ}' - \text{נ}'')\frac{du}{dx} = 0 \qquad (39)$$

$$(\text{ג}' - \text{ג}'')\frac{du}{dz} + (\text{מ}' - \text{מ}'')\frac{du}{dx} + (\text{ל}' - \text{ל}'')\frac{du}{dy} = 0$$

As an example of the formulæ, we shall suppose the bodies to be in contact along an indefinite plane, which we shall take for the plane $(x, y)$. The surface $u = f(x, y, z) = 0$ becomes for this case $z = 0$, and consequently we shall have to introduce the conditions,

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 1.$$

The equations of condition to be satisfied at the limits become, therefore,

$$\text{מ}' - \text{מ}'' = 0, \quad \text{ל}' - \text{ל}'' = 0, \quad \text{ג}' - \text{ג}'' = 0, \tag{40}$$

i. e. the three quantities ג, ל, מ, *retain their values* in passing from one medium to another. In order to obtain a more geometrical idea of the nature of these equations of condition, in the particular case of plane waves, we shall assume

$$\text{א} = \frac{d^2p}{dt^2}, \quad \text{ב} = \frac{d^2q}{dt^2}, \quad \text{ג} = \frac{d^2r}{dt^2}; \qquad \text{ל} = \frac{d^2f}{dt^2}, \quad \text{מ} = \frac{d^2g}{dt^2}, \quad \text{נ} = \frac{d^2h}{dt^2},$$

where $(p, q, r)$ are the components of a transversal $\tau_{,}$, and $(f, g, h)$ of another transversal $\tau_{,,}$, so that we have

$$p = \tau_{,}\cos\lambda_{,}, \quad q = \tau_{,}\cos\mu_{,}, \quad r = \tau_{,}\cos\nu_{,},$$
$$f = \tau_{,,}\cos\lambda_{,,}, \quad g = \tau_{,,}\cos\mu_{,,}, \quad h = \tau_{,,}\cos\nu_{,,}.$$

These quantities $(\tau_{,}, \tau_{,,})$ may be called the first and second *transversals* of the real molecular vibration which takes place along $(\alpha, \beta, \gamma)$, and is equal to $\tau$; and they have a striking analogy to the line that Professor M'Cullagh calls a secondary transversal in his theory of light; but there are two such transversals in the theory of elastic solids, while there is only one in the theory of light. It should be remembered that the equations of condition at the limits (32–40) are perfectly general, and independent of the particular integral for plane-waves; but the geometrical constructions, which I proceed to investigate, belong only to the particular integral. Previously to determining the meaning of equations (40) it will be useful to establish a theorem relating to surfaces of the second order. Let the surface be

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{D}yz + 2\mathrm{E}xz + 2\mathrm{F}xy = 1,$$

then we shall have

$$\left.\begin{array}{c}\dfrac{\cos(p,r')}{pr} \\ \dfrac{\cos(p',r)}{p'r'}\end{array}\right\} = \mathrm{A}\cos\alpha\cos\alpha' + \mathrm{B}\cos\beta\cos\beta' + \mathrm{C}\cos\gamma\cos\gamma' + \mathrm{D}(\cos\beta\cos\gamma' + \cos\gamma\cos\beta') + \mathrm{E}(\cos\gamma\cos\alpha' + \cos\alpha\cos\gamma') + \mathrm{F}(\cos\alpha\cos\beta' + \cos\beta\cos\alpha'), \tag{41}$$

where $(p, p', r, r')$ are the perpendiculars let fall on any two tangent planes and the corresponding radii vectores, and $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ the directions of $(r, r')$. This expression for $\frac{\cos(p,r')}{pr} = \frac{\cos(p',r)}{p'r'}$, in terms of $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$, may be proved as follows: the tangent plane to the surface is

$$(\mathrm{A}x + \mathrm{F}y + \mathrm{E}z)x' + (\mathrm{B}y + \mathrm{D}z + \mathrm{F}x)y' + (\mathrm{C}z + \mathrm{E}x + \mathrm{D}y)z' = 1;$$

therefore, if $(\lambda, \mu, \nu)$ denote the direction of the perpendicular, we shall have

$$\begin{aligned}
\cos\lambda &= pr(\mathrm{A}\cos\alpha + \mathrm{F}\cos\beta + \mathrm{E}\cos\gamma)\\
\cos\mu &= pr(\mathrm{B}\cos\beta + \mathrm{D}\cos\gamma + \mathrm{F}\cos\alpha)\\
\cos\nu &= pr(\mathrm{C}\cos\gamma + \mathrm{E}\cos\alpha + \mathrm{D}\cos\beta)
\end{aligned}$$

therefore

$$\frac{\cos\lambda\cos\alpha' + \cos\mu\cos\beta' + \cos\nu\cos\gamma'}{pr} = \frac{\cos\lambda'\cos\alpha + \cos\mu'\cos\beta + \cos\nu'\cos\gamma}{p'r'}$$

$$= \mathrm{A}\cos\alpha\cos\alpha' + \mathrm{B}\cos\beta\cos\beta' + \mathrm{C}\cos\gamma\cos\gamma'$$

$$+\mathrm{D}(\cos\beta\cos\gamma' + \cos\gamma\cos\beta') + \mathrm{E}(\cos\gamma\cos\alpha' + \cos\alpha\cos\gamma') + \mathrm{F}(\cos\alpha\cos\beta' + \cos\beta\cos\alpha');$$

which is equivalent to (41).

If now we substitute for $(\xi, \eta, \zeta)$ their values,

$$\xi = \tau\cos\alpha.\cos\phi, \quad \eta = \tau\cos\beta.\cos\phi, \quad \zeta = \tau\cos\gamma.\cos\phi,$$

$$\phi = \frac{2\pi}{\lambda}(lx + my + nz - \nu t)$$

in the equations

$$\frac{d^2p}{dt^2} = \aleph = \mathrm{A}\frac{d\xi}{dx} + \mathrm{N}\frac{d\eta}{dy} + \mathrm{M}\frac{d\zeta}{dz} + \alpha_1\mathrm{U} + \alpha_2\mathrm{V} + \alpha_3\mathrm{W}$$

$$\frac{d^2q}{dt^2} = \beth = \mathrm{B}\frac{d\eta}{dy} + \mathrm{L}\frac{d\zeta}{dz} + \mathrm{N}\frac{d\xi}{dx} + \beta_1\mathrm{U} + \beta_2\mathrm{V} + \beta_3\mathrm{W}$$

$$\frac{d^2r}{dt^2} = \gimel = \mathrm{C}\frac{d\zeta}{dz} + \mathrm{M}\frac{d\xi}{dx} + \mathrm{L}\frac{d\eta}{dy} + \gamma_1\mathrm{U} + \gamma_2\mathrm{V} + \gamma_3\mathrm{W}$$

$$\frac{d^2f}{dt^2} = \text{ל} = \alpha_1\frac{d\xi}{dx} + \beta_1\frac{d\eta}{dy} + \gamma_1\frac{d\zeta}{dz} + \mathrm{LU} + \gamma_3\mathrm{V} + \beta_2\mathrm{W}$$

$$\frac{d^2g}{dt^2} = \text{ט} = \alpha_2\frac{d\xi}{dx} + \beta_2\frac{d\eta}{dy} + \gamma_2\frac{d\zeta}{dz} + \gamma_3\mathrm{U} + \mathrm{MV} + \alpha_1\mathrm{W}$$

$$\frac{d^2h}{dt^2} = \text{כ} = \alpha_3\frac{d\xi}{dx} + \beta_3\frac{d\eta}{dy} + \gamma_3\frac{d\zeta}{dz} + \beta_2\mathrm{U} + \alpha_1\mathrm{V} + \mathrm{NW}$$

we shall obtain, by integrating twice with regard to *t*, the following expressions,

$$\begin{array}{ll} p = \tau_{\prime}\cos\lambda_{\prime} = \mathrm{I}.h_{\prime}\sin\phi & f = \tau_{\prime\prime}\cos\lambda_{\prime\prime} = \mathrm{I}.k_{\prime}\sin\phi \\ q = \tau_{\prime}\cos\mu_{\prime} = \mathrm{I}.h_{\prime\prime}\sin\phi & g = \tau_{\prime\prime}\cos\mu_{\prime\prime} = \mathrm{I}.k_{\prime\prime}\sin\phi \\ r = \tau_{\prime}\cos\nu_{\prime} = \mathrm{I}.h_{\prime\prime\prime}\sin\phi & h = \tau_{\prime\prime}\cos\nu_{\prime\prime} = \mathrm{I}.k_{\prime\prime\prime}\sin\phi \end{array} \qquad (42)$$

where we have $\mathrm{I} = \frac{\mathrm{T}^2\tau}{2\pi\lambda}$; and

$$\begin{aligned} h_{\prime} = {} & \mathrm{A}.l\cos\alpha + \mathrm{N}.m\cos\beta + \mathrm{M}.n\cos\gamma \\ & + \alpha_1(m\cos\gamma + n\cos\beta) + \alpha_2(n\cos\alpha + l\cos\gamma) + \alpha_3(l\cos\beta + m\cos\alpha) \end{aligned}$$

$$\begin{aligned} h_{\prime\prime} = {} & \mathrm{B}.m\cos\beta + \mathrm{L}.n\cos\gamma + \mathrm{N}.l\cos\alpha \\ & + \beta_1(m\cos\gamma + n\cos\beta) + \beta_2(n\cos\alpha + l\cos\gamma) + \beta_3(l\cos\beta + m\cos\alpha) \end{aligned}$$

$$\begin{aligned} h_{\prime\prime\prime} = {} & \mathrm{C}.n\cos\gamma + \mathrm{M}.l\cos\alpha + \mathrm{L}.m\cos\beta \\ & + \gamma_1(m\cos\gamma + n\cos\beta) + \gamma_2(n\cos\alpha + l\cos\gamma) + \gamma_3(l\cos\beta + m\cos\alpha) \end{aligned}$$

$$\begin{aligned} k_{\prime} = {} & \alpha_1.l\cos\alpha + \beta_1.m\cos\beta + \gamma_1.n\cos\gamma \\ & + \mathrm{L}(m\cos\gamma + n\cos\beta) + \gamma_3(n\cos\alpha + l\cos\gamma) + \beta_2(l\cos\beta + m\cos\alpha) \end{aligned}$$

$$\begin{aligned} k_{\prime\prime} = {} & \alpha_2.l\cos\alpha + \beta_2.m\cos\beta + \gamma_2.n\cos\gamma \\ & + \gamma_3(m\cos\gamma + n\cos\beta) + \mathrm{M}(n\cos\alpha + l\cos\gamma) + \alpha_1(l\cos\beta + m\cos\alpha) \end{aligned}$$

$$\begin{aligned} k_{\prime\prime\prime} = {} & \alpha_3.l\cos\alpha + \beta_3.m\cos\beta + \gamma_3.n\cos\gamma \\ & + \beta_2(m\cos\gamma + n\cos\beta) + \alpha_1(n\cos\alpha + l\cos\gamma) + \mathrm{N}(l\cos\beta + m\cos\alpha) \end{aligned}$$

We have, therefore, the following equations to determine the direction and magnitude of the two transversals:

$$\begin{array}{ll} h_{\prime\prime}\cos\nu_{\prime} - h_{\prime\prime\prime}\cos\mu_{\prime} = 0 & k_{\prime\prime}\cos\nu_{\prime\prime} - k_{\prime\prime\prime}\cos\mu_{\prime\prime} = 0 \\ h_{\prime\prime\prime}\cos\lambda_{\prime} - h_{\prime}\cos\nu_{\prime} = 0 & k_{\prime\prime\prime}\cos\lambda_{\prime\prime} - k_{\prime}\cos\nu_{\prime\prime} = 0 \\ h_{\prime}\cos\mu_{\prime} - h_{\prime\prime}\cos\lambda_{\prime} = 0 & k_{\prime}\cos\mu_{\prime\prime} - k_{\prime\prime}\cos\lambda_{\prime\prime} = 0 \end{array} \qquad (43)$$

$$\tau_{\prime} = \mathrm{I}.\sqrt{h_{\prime}^2 + h_{\prime\prime}^2 + h_{\prime\prime\prime}^2}\,\sin\phi \qquad \tau_{\prime\prime} = \mathrm{I}.\sqrt{k_{\prime}^2 + k_{\prime\prime}^2 + k_{\prime\prime\prime}^2}\,\sin\phi$$

If we suppose the six fixed ellipsoids (29) constructed, with any point in the plane of separation for common centre; and make $(p_{\prime}, p_{\prime\prime}, p_{\prime\prime\prime}, \pi_{\prime}, \pi_{\prime\prime}, \pi_{\prime\prime\prime})$ the perpendiculars on tangent planes drawn at the six points where the normal of the incident wave-plane pierces the ellipsoids; the intercepts of the normal itself being $(r_{,}, r_{,,}, r_{,,,}, \rho_{,}, \rho_{,,}, \rho_{,,,})$; and let $(a)$ denote the direction $(\alpha, \beta, \gamma)$ in which the real molecular vibration takes place; then we shall have the following geometrical meanings for the quantities ($h$, &c., $k$, &c.),

$$\begin{array}{ll} h_{,} = \dfrac{\cos(ap_{,})}{p_{,}r_{,}} & k_{,} = \dfrac{\cos(a\pi_{,})}{\pi_{,}\rho_{,}} \\ h_{,,} = \dfrac{\cos(ap_{,,})}{p_{,,}r_{,,}} & k_{,,} = \dfrac{\cos(a\pi_{,,})}{\pi_{,,}\rho_{,,}} \\ h_{,,,} = \dfrac{\cos(ap_{,,,})}{p_{,,,}r_{,,,}} & k_{,,,} = \dfrac{\cos(a\pi_{,,,})}{\pi_{,,,}\rho_{,,,}} \end{array} \tag{44}$$

Hence the equations for determining the direction of the two transversals become finally,

$$\begin{array}{l} p_{,,,}r_{,,,}.\cos(ap_{,,})\cos\nu_{,} - p_{,,}r_{,,}.\cos(ap_{,,,})\cos\mu_{,} = 0 \\ p_{,}r_{,}.\cos(ap_{,,,})\cos\lambda_{,} - p_{,,,}r_{,,,}.\cos(ap_{,})\cos\nu_{,} = 0 \\ p_{,,}r_{,,}.\cos(ap_{,})\cos\mu_{,} - p_{,}r_{,}.\cos(ap_{,,})\cos\lambda_{,} = 0 \\ \\ \pi_{,,,}\rho_{,,,}.\cos(a\pi_{,,})\cos\nu_{,,} - \pi_{,,}\rho_{,,}.\cos(a\pi_{,,,})\cos\mu_{,,} = 0 \\ \pi_{,}\rho_{,}.\cos(a\pi_{,,,})\cos\lambda_{,,} - \pi_{,,,}\rho_{,,,}.\cos(a\pi_{,})\cos\nu_{,,} = 0 \\ \pi_{,,}\rho_{,,}.\cos(a\pi_{,})\cos\mu_{,,} - \pi_{,}\rho_{,}.\cos(a\pi_{,,})\cos\lambda_{,,} = 0 \end{array} \tag{45}$$

The two transversals are thus completely determined in magnitude and direction, and are to be conceived as accompanying the *real* vibration in its progress, depending upon its direction and velocity; and the equations of condition (40) at the limits are expressed by saying that, in passing from one medium into another, the *first* transversal preserves its value *perpendicular* to the separating surface, while the *second* transversal preserves its value *parallel* to the same surface: the *real* vibration preserving its value both perpendicular and parallel to the surface of separation, since the molecules composing it may be viewed as belonging indifferently to either medium. It is important to observe that we have thus as many conditions as unknown quantities; for, the incident vibration being given, we have to determine three refracted and three reflected vibrations; the directions of these are known by means of the surface of wave-slowness, and the quantities which are to be determined are the ***six intensities*** of the reflected and refracted vibrations. Now we have six conditions to determine these six quantities; these conditions are—the conservation of the real vibration in passing from one medium into another—the conservation of the perpendicular component of the first transversal—and the conservation of the second transversal parallel to the plane of separation: the equations of condition given by these considerations are,

$$\begin{aligned} &\xi' = \xi'', \qquad \eta' = \eta'', \qquad \zeta' = \zeta'', \\ &\gimel' = \gimel'', \qquad \text{ל}' = \text{ל}'', \qquad \text{מ}' = \text{מ}''. \end{aligned} \tag{46}$$

If the vibration pass from a homogeneous solid into one of crystalline structure, there will be three refracted and two reflected waves; and the unknown quantities will be the *five intensities and one direction* in the homogeneous body, which is not given by the wave-surface; or, *vice versâ*, if the vibration pass from the crystalline into the homogeneous body, there will be three reflected and two refracted waves, and the six unknown quantities will be the same as before. Also, if two homogeneous bodies be superposed, it will be shown that there are two reflected and two refracted waves, and the unknown quantities are *four intensities and two directions*. In fact, whatever be the nature of the superposed bodies, the unknown quantities will be six in number, so that they can be completely determined by the six equations of condition.

We have hitherto considered the differential equations arising from $V_1$ expressed in its most general form, and although the relations (15) might have been introduced, yet the simplification produced by them would be very slight; the principal effect of these equations being to modify the equation of wave-slowness, and it appears very difficult to obtain any other relations among the coefficients of $V_1$, which should simplify to any important extent the equations of motion.

I shall now discuss a few particular cases of solid bodies, in order to show the manner in which the general equations already found should be modified when applied to such cases.

I. Let us suppose the solid to have its molecular constitution such, that it is symmetrically arranged round three rectangular planes, i. e. the molecular forces are in every respect similar in each of the eight regions into which the three planes drawn at any point divide the body. It is easily seen that in such a medium the coefficients $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3)$ of the function $\mathrm{V}_1$ are all zero (11), and that consequently $\mathrm{V}_1$ is reduced to the form,

$$2\mathrm{V}_1 = \left(\mathrm{A}\left(\frac{d\xi}{dx}\right)^2 + \mathrm{B}\left(\frac{d\eta}{dy}\right)^2 + \mathrm{C}\left(\frac{d\zeta}{dz}\right)^2\right) + (\mathrm{LU}^2 + \mathrm{MV}^2 + \mathrm{NW}^2)$$
$$+ 2\left(\mathrm{L}\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz} + \mathrm{M}\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz} + \mathrm{N}\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)$$

We shall also have the equations (26) reduced to

$$\begin{aligned}
\epsilon\frac{d^2\xi}{dt^2} &= \mathrm{A}\frac{d^2\xi}{dx^2} + \mathrm{N}\frac{d^2\xi}{dy^2} + \mathrm{M}\frac{d^2\xi}{dz^2} + 2\left(\mathrm{N}\frac{d^2\eta}{dxdy} + \mathrm{M}\frac{d^2\zeta}{dxdz}\right) \\
\epsilon\frac{d^2\eta}{dt^2} &= \mathrm{B}\frac{d^2\eta}{dy^2} + \mathrm{L}\frac{d^2\eta}{dz^2} + \mathrm{N}\frac{d^2\eta}{dx^2} + 2\left(\mathrm{L}\frac{d^2\zeta}{dydz} + \mathrm{N}\frac{d^2\xi}{dxdy}\right) \qquad (47) \\
\epsilon\frac{d^2\zeta}{dt^2} &= \mathrm{C}\frac{d^2\zeta}{dz^2} + \mathrm{M}\frac{d^2\zeta}{dx^2} + \mathrm{L}\frac{d^2\zeta}{dy^2} + 2\left(\mathrm{M}\frac{d^2\xi}{dxdz} + \mathrm{L}\frac{d^2\eta}{dydz}\right)
\end{aligned}$$

The equation of the surface of wave-slowness will still be represented by (31), in which we must now give to (P, Q, R, F, G, H) the following values,

$$\begin{aligned}
\mathrm{P} &= \mathrm{A}x^2 + \mathrm{N}y^2 + \mathrm{M}z^2 & \mathrm{F} &= 2\mathrm{L}yz \\
\mathrm{Q} &= \mathrm{B}y^2 + \mathrm{L}z^2 + \mathrm{N}x^2 & \mathrm{G} &= 2\mathrm{M}xz \qquad (48) \\
\mathrm{R} &= \mathrm{C}z^2 + \mathrm{M}x^2 + \mathrm{L}y^2 & \mathrm{H} &= 2\mathrm{N}xy
\end{aligned}$$

and its traces on the principal planes of symmetry are given by the following equations:

$$\begin{aligned}
(\mathrm{L}y^2 + \mathrm{M}x^2 - 1)\left[(\mathrm{A}x^2 + \mathrm{N}y^2 - 1)(\mathrm{B}y^2 + \mathrm{N}x^2 - 1) - 4\mathrm{N}^2x^2y^2\right] &= 0 \\
(\mathrm{N}x^2 + \mathrm{L}z^2 - 1)\left[(\mathrm{C}z^2 + \mathrm{M}x^2 - 1)(\mathrm{A}x^2 + \mathrm{M}z^2 - 1) - 4\mathrm{M}^2x^2z^2\right] &= 0 \qquad (49) \\
(\mathrm{M}z^2 + \mathrm{N}y^2 - 1)\left[(\mathrm{B}y^2 + \mathrm{L}z^2 - 1)(\mathrm{C}z^2 + \mathrm{L}y^2 - 1) - 4\mathrm{L}^2y^2z^2\right] &= 0
\end{aligned}$$

for it appears immediately from the equation of the surface, that its traces are

$$\begin{aligned}
(\mathrm{R}-1)\left[(\mathrm{P}-1)(\mathrm{Q}-1) - \mathrm{H}^2\right] &= 0 & z &= 0 \\
(\mathrm{Q}-1)\left[(\mathrm{P}-1)(\mathrm{R}-1) - \mathrm{G}^2\right] &= 0 & y &= 0 \\
(\mathrm{P}-1)\left[(\mathrm{Q}-1)(\mathrm{R}-1) - \mathrm{F}^2\right] &= 0 & x &= 0
\end{aligned}$$

Hence the traces on the principal planes of the surface of *wave-slowness* consist of an ellipse and a curve of the fourth degree; which establishes an analogy between the case of solid elastic bodies and light, where the corresponding traces are a circle and an ellipse.

The surface of *wave-slowness* of symmetrical elastic bodies possesses singular points, of the nature of *nodes*, in its principal planes, for the equation of the surface may be put under any of the forms

$$\begin{aligned} \phi_1 + z^2.\psi_1 &= 0 \\ \phi_2 + y^2.\psi_2 &= 0 \\ \phi_3 + x^2.\psi_3 &= 0 \end{aligned}$$

where the values of $\phi_1$, $\phi_2$, $\phi_3$ are

$$\begin{aligned} \phi_1 &= (\text{R}-1)\,[(\text{P}-1)\,(\text{Q}-1) - \text{H}^2] \\ \phi_2 &= (\text{Q}-1)\,[(\text{P}-1)\,(\text{R}-1) - \text{G}^2] \\ \phi_3 &= (\text{P}-1)\,[(\text{Q}-1)\,(\text{R}-1) - \text{F}^2] \end{aligned}$$

Now, if the equation of a surface be

$$u = \phi + z^2.\psi = 0,$$

it may be shown that, if the surface $\phi = 0$ has any singular points in the plane $z = 0$, that these are also singular points in the surface $u = 0$; for the conditions for singular points are

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0;$$

and if $z = 0$, the part of $u$ of the form $(z^2.\psi)$ will disappear from the three equations,

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0,$$

which will then be of the form

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0,$$

which are the conditions for singular points in the surface $\phi = 0$. Hence, if any singular points of the surface $\phi = 0$ exist in the plane $z = 0$, these will be also singular points of the surface $u = 0$. Applying this principle to the three forms of the surface of wave-slowness just given, it will appear that the intersections of

$$\text{R} - 1 = 0, \quad (\text{P} - 1)(\text{Q} - 1) - \text{H}^2 = 0, \quad z = 0,$$

will be singular points of the surface of wave-slowness in the plane $(x, y)$; also

$$\text{Q} - 1 = 0, \quad (\text{P} - 1)(\text{R} - 1) - \text{G}^2 = 0, \quad y = 0,$$

will give singular points in the plane $(x, z)$; and

$$\text{P} - 1 = 0, \quad (\text{Q} - 1)(\text{R} - 1) - \text{F}^2 = 0, \quad x = 0,$$

will give singular points in the plane $(y, z)$.

As the traces in the principal planes consist of a curve of the second and one of the fourth degree, we shall have in general eight points of intersection, real or imaginary, and therefore the surface of wave-slowness should have twenty-four singular points; but as it is only the *real* points of intersection that produce any effect in the physical problem, it becomes of importance to ascertain the number and position of the real singular points.

I shall first prove that the curve of the fourth degree consists of two ovals, lying one inside the other, and not having any point in common. The equation of the curve on the plane $(x, y)$ is

$$(\text{A}x^2 + \text{N}y^2 - 1)(\text{B}y^2 + \text{N}x^2 - 1) - 4\text{N}^2x^2y^2 = 0,$$

and its polar equation is

$$\left(\text{A}\cos^2\alpha + \text{N}\sin^2\alpha - \frac{1}{\rho^2}\right)\left(\text{B}\sin^2\alpha + \text{N}\cos^2\alpha - \frac{1}{\rho^2}\right) - 4\text{N}^2\sin^2\alpha\cos^2\alpha = 0,$$

which is a quadratic equation with respect to $\frac{1}{\rho^2}$; the condition necessary for *equal roots* is

$$[(\text{A} + \text{N})\cos^2\alpha + (\text{B} + \text{N})\sin^2\alpha]^2$$
$$= 4(\text{A}\cos^2\alpha + \text{N}\sin^2\alpha)(\text{B}\sin^2\alpha + \text{N}\cos^2\alpha) - 16\text{N}^2\sin^2\alpha\cos^2\alpha;$$

and, if two radii vectores of the curve coincide, i. e. if the curve have a double point, this equation of condition must give a *real* value for $\alpha$; arranging it with respect to $\tan\alpha$, it becomes

$$(\text{B}-\text{N})^2.\tan^4 a+[2(\text{A}+\text{N})(\text{B}+\text{N})-4\text{AB}+12\text{N}^2]\tan^2 a+(\text{A}-\text{N})^2=0;$$

or, assuming

$$\omega=(\text{A}-\text{N})(\text{B}-\text{N})-4\text{N}^2,$$

$$(\text{B}-\text{N})^2.\tan^4 a-2(\omega-4\text{N}^2).\tan^2 a+(\text{A}-\text{N})^2=0.$$

Now, in order that this equation should give *real* values for $\tan a$, it must give *real and positive* values for $\tan^2 a$; but it can be shown that the roots of this quadratic equation, if real, are *negative*, and, consequently, that no real value exists for $\tan a$.

The equation solved for $\tan^2 a$ may be put under the form

$$\tan^2 a.(\text{B}-\text{N})^2=(\omega-4\text{N}^2)\pm\sqrt{(\omega-4\text{N}^2)^2-(\omega+4\text{N}^2)^2},$$

or

$$\tan^2 a.(\text{B}-\text{N})^2=(\omega-4\text{N}^2)\pm\sqrt{-16\text{N}^2.\omega}.$$

These equations show that the condition for real values of $\tan^2 a$ is, that $\omega$ must be *negative* or *zero;* and that in either case $\tan^2 a$ must be *negative.* Hence we have proved, in general, that the two branches of the curve cannot have a real point of intersection. The same thing is true of the curves of the fourth degree in the other principal planes.

The geometrical meaning of the condition $\omega=0$ is, that the curve of the fourth degree, in each plane, should be the product of an ellipse and circle whose equations are

$$\text{A}x^2+\text{B}y^2-1=0,\qquad \text{N}x^2+\text{N}y^2-1=0,\qquad \text{in plane } (x,y).$$
$$\text{B}y^2+\text{C}z^2-1=0,\qquad \text{L}y^2+\text{L}z^2-1=0,\qquad \text{in plane } (y,z).$$
$$\text{C}z^2+\text{A}x^2-1=0,\qquad \text{M}z^2+\text{M}x^2-1=0,\qquad \text{in plane } (x,z).$$

The curve of the fourth degree consists of two branches, each of which cuts at right angles, the axes of coordinates, in four points, and the semiaxes of one branch are $\frac{1}{\sqrt{\text{A}}}$, $\frac{1}{\sqrt{\text{B}}}$, and the semiaxes of the other branch are equal, and each $\frac{1}{\sqrt{\text{N}}}$, and similarly for the other coordinate planes.

The equation of the ellipse in the plane $(x,y)$ being

$$\text{L}y^2+\text{M}x^2-1=0,$$

this curve will evidently cut the oval, whose semiaxes are $\frac{1}{\sqrt{\text{N}}}$, if N be intermediate in value between L and M; supposing, therefore, that M is the mean of the three quantities, L, M, N, the ellipse in the plane $(x, z)$ will cut the branch of the oval, whose semiaxes are $\frac{1}{\sqrt{\text{M}}}$; and in the other coordinate planes it will lie, in one case, completely outside the oval; and, in the other case, completely inside the oval. Hence there are always *at least* four singular points on the surface of wave-slowness; whether there be more singular points will depend on the relative magnitude of A, B, C, compared with L, M, N; if we assume (as seems probable from its being true in homogeneous solids) that A, B, C are greater than L, M, N, then the ovals whose semiaxes are $\frac{1}{\sqrt{\text{A}}}, \frac{1}{\sqrt{\text{B}}}, \frac{1}{\sqrt{\text{C}}}$ will lie completely inside both the ellipses and the other ovals, whose semiaxes are equal to $\frac{1}{\sqrt{\text{L}}}, \frac{1}{\sqrt{\text{M}}}, \frac{1}{\sqrt{\text{N}}}$. The surface of wave-slowness will therefore consist of three sheets; one, whose semiaxes are $\frac{1}{\sqrt{\text{A}}}, \frac{1}{\sqrt{\text{B}}}, \frac{1}{\sqrt{\text{C}}}$, isolated, and lying inside the other two sheets; and the other two sheets, having four points in common, like Fresnel's wave-surface, and piercing the axis of $x$ in the points $\frac{1}{\sqrt{\text{M}}}, \frac{1}{\sqrt{\text{N}}}$, the axis of $y$ in $\frac{1}{\sqrt{\text{L}}}, \frac{1}{\sqrt{\text{N}}}$, and axis of $z$ in $\frac{1}{\sqrt{\text{L}}}, \frac{1}{\sqrt{\text{M}}}$.

I shall now proceed to discuss some particular cases of the motion, and to explain the phenomena which arise from the existence of nodes in the surface of wave-slowness.

Let a plane wave move through the body, perpendicular to $z$; the auxiliary ellipsoid becomes, in this case,

$$\text{M}x^2 + \text{L}y^2 + \text{C}z^2 = 1,$$

because $l = 0$, $m = 0$, $n = 1$; hence the three possible vibrations of the molecules will be parallel to the axes of symmetry, and two of them will lie in the wave-plane at right angles to each other, and the other will be perpendicular to the wave-plane; and the three velocities of propagation will be proportional to

$\sqrt{\text{C}}$, $\sqrt{\text{L}}$, $\sqrt{\text{M}}$; $\sqrt{\text{C}}$ belonging to the wave whose vibrations are normal to its plane, and $\sqrt{\text{L}}$, $\sqrt{\text{M}}$, belonging to the waves in which the vibrations are in the plane of the wave. Similar conclusions will, of course, hold for waves parallel to the other planes of symmetry. It may be observed that, in the case of these waves, the direction in which the motion is propagated through the solid (which corresponds to the *ray* in the theory of light) coincides with the perpendicular on wave plane; for it may be proved that the three axes of co-ordinates pierce the surface of wave-slowness in eighteen points, at each of which the tangent plane is perpendicular to the radius vector; for the equation of the tangent plane is

$$\frac{du}{dx}(x'-x)+\frac{du}{dy}(y'-y)+\frac{du}{dz}(z'-z)=0,$$

$u=0$ being the equation of the surface; but, if we make $x=0$, $y=0$, it is easily seen that $\frac{du}{dx}=0$, $\frac{du}{dy}=0$, and consequently the tangent planes at the points where $z$ pierces the surface are given by the equation

$$\frac{du}{dz}(z'-z)=0,$$

which denotes a plane parallel to $(x, y)$.

In general, when a wave-plane passes in any direction through the solid, the vibrations of the molecules for which such a direction of wave-plane is possible will not lie in the wave-plane; but there are some cases besides the case of waves parallel to the principal planes, in which the vibrations are, one normal to the wave-plane, and the other two lying in the wave-plane.

Let us consider the case of a wave-plane parallel only to one of the principal *axes* $(z)$; the auxiliary ellipsoid becomes for this case

$$\text{P}_1x^2+\text{Q}_1y^2+\text{R}_1z^2+2\text{H}_1xy=1,$$

where

$$\begin{aligned} \text{P}_1 &= \text{A}l^2+\text{N}m^2, & \text{H}_1 &= 2\text{N}lm, \\ \text{Q}_1 &= \text{B}m^2+\text{N}l^2, & &\text{because } n=0, \\ \text{R}_1 &= \text{M}l^2+\text{L}m^2. \end{aligned}$$

This equation proves that for every position of waves parallel to an axis, there is one vibration in the wave-plane parallel to the axis.

Now, the equation which determines the axes of the ellipsoid lying in the plane $(x, y)$ is

$$\mathrm{H}_1 \tan^2\phi + (\mathrm{P}_1 - \mathrm{Q}_1) . \tan\phi - \mathrm{H}_1 = 0.$$

If we make $\tan\phi = \frac{m}{l}$, we shall have the condition necessary to be fulfilled, in order that the other two vibrations may lie, one in the wave-plane, and the other normal to it; this condition is

$$lm[(\mathrm{A} - 3\mathrm{N})l^2 - (\mathrm{B} - 3\mathrm{N})m^2] = 0.$$

The factors $l = 0$, or $m = 0$, give the axes $(x, y)$, which have been already discussed, and the third factor gives the equation

$$\tan^2\phi = \frac{\mathrm{A} - 3\mathrm{N}}{\mathrm{B} - 3\mathrm{N}},$$

which shows that there are two lines in the plane $(x, y)$, making equal angles with the axis of $x$, such that if a wave-plane pass through the solid perpendicular to either of them, the three vibrations possible for that direction of wave-plane will be two transversal and one normal.

Thus the existence of nine directions in the solid, for which the vibrations are in the wave-plane and normal to it, has been proved; viz., the three axes of symmetry, and two directions in each plane of symmetry.

To show the effect produced by the existence of nodes in the surface of wave-slowness, it will be necessary to consider a wave in its passage from one solid to another; the construction for the refracted wave is as follows: describe the surfaces of wave-slowness (A, B) for both media, having a common centre in the plane which separates the two media; produce the normal to the incident wave to meet the corresponding sheet of its own surface (A), and from the point of intersection let fall a perpendicular on the separating plane, this perpendicular will in general pierce the surface (B) in three points; and the corresponding radii vectores will be the normals to the three refracted *waves*, the perpendiculars on the corresponding tangent planes being the directions of the refracted *rays*. Suppose the perpendicular should pierce the surface (B) in a node, the line joining the centre of (B) with the node will be the normal to refracted wave; but there will be an infinite number of rays, which will form the sides of a cone of the second degree, having its vertex at the centre of the surface of wave-slowness, and reciprocal to the tangent cone at the node. Also, there may be one ray, and an infinite number of waves; for if we consider that there are four tangent planes of the surface (B), which touch the surface along an ellipse, it is evident that there might be a cone of refracted wave normals, of the second degree, whose base is one of the ellipses of contact, while there would only be one ray, whose direction is the perpendicular on tangent plane: to find, in this case, the cone of incident waves which will be refracted into a single ray, we must project, by perpendiculars to the plane of separation, the ellipse of contact of surface (B) upon (A); then the cone whose base is the projection will be the cone of incident wave normals; while the cone whose base is the ellipse will be the cone of refracted wave normals, and the unique direction of the ray will be the perpendicular to the plane of the ellipse.

II. If the molecules of the body be arranged symmetrically round one axis, the differential equations of motion will still be the same as (47), with the following relations among the constants:

$$\mathrm{A} = \mathrm{B} = 3\mathrm{N}, \qquad \mathrm{L} = \mathrm{M}.$$

In these equations the axis of $z$ is the axis of symmetry; two of these relations are evident, $\mathrm{A} = \mathrm{B}$, $\mathrm{L} = \mathrm{M}$; and the third may be thus proved. The function $(\mathrm{F}_1)$, in the coefficients (11), is in general a function of $(\rho, \theta, \phi)$; but since, in the present case, everything is similar for all planes passing through the axis of symmetry, $\mathrm{F}_1$ will not be a function of $\phi$, therefore the expressions for $\mathrm{A}$, $\mathrm{B}$, $\mathrm{N}$ may be integrated once with respect to $\phi$,

$$\mathrm{A} = \mathrm{B} = 2\iiint \mathrm{F}_1 \cos^4\alpha\, d\omega, \qquad \mathrm{N} = 2\iiint \mathrm{F}_1 \cos^2\alpha \cos^2\beta\, d\omega;$$

or, substituting for $\cos\alpha$, $\cos\beta$, their values, and assuming

$$\Omega = 2\iint \mathrm{F}_1 \sin^5\theta \rho^4 d\rho d\theta,$$

we shall have

$$\mathrm{A} = \mathrm{B} = \int_0^{2\pi} \Omega.\cos^4\phi d\phi, \qquad \mathrm{N} = \int_0^{2\pi} \Omega.\sin^2\phi \cos^2\phi d\phi,$$

where $\Omega$ is independent of $\phi$; and, finally, integrating these two expressions, we shall obtain

$$\mathrm{A} = \mathrm{B} = 3\mathrm{N}.$$

These relations introduce corresponding simplifications into the surface of wave-slowness, which will evidently become a surface of revolution round the axis of $z$. There will be no conical refraction for any direction of wave-transmission; but, since two sheets of the surface touch each other on the axis of $z$, there will be only two velocities of waves parallel to $(x, y)$, while there will be three velocities in every other direction.

III. If the medium be perfectly homogeneous and uncrystalline we shall still use the equations (47) with the additional relations,

$$\mathrm{A} = \mathrm{B} = \mathrm{C}, \quad \mathrm{L} = \mathrm{M} = \mathrm{N}, \quad \mathrm{A} = 3\mathrm{N},$$

which reduce them to the following form:

$$\begin{aligned}
\epsilon\frac{d^2\xi}{dt^2} &= \mathrm{N}\left(3\frac{d^2\xi}{dx^2} + \left(\frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2}\right) + 2\left(\frac{d^2\eta}{dxdy} + \frac{d^2\zeta}{dxdz}\right)\right)\\
\epsilon\frac{d^2\eta}{dt^2} &= \mathrm{N}\left(3\frac{d^2\eta}{dy^2} + \left(\frac{d^2\eta}{dz^2} + \frac{d^2\eta}{dx^2}\right) + 2\left(\frac{d^2\zeta}{dydz} + \frac{d^2\xi}{dxdy}\right)\right) \qquad (50)\\
\epsilon\frac{d^2\zeta}{dt^2} &= \mathrm{N}\left(3\frac{d^2\zeta}{dz^2} + \left(\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2}\right) + 2\left(\frac{d^2\xi}{dxdz} + \frac{d^2\eta}{dydz}\right)\right)
\end{aligned}$$

These are the differential equations of motion of a solid homogeneous and uncrystalline body, and were given first by Navier in the memoir already alluded to.

Referring to the general form of $\mathrm{v}_1$, it will readily appear that, for the case at present considered, it becomes

$$\begin{aligned}
2\mathrm{v}_1 = \mathrm{N}\Bigg(3\left(\frac{d\xi^2}{dx^2} + \frac{d\eta^2}{dy^2} + \frac{d\zeta^2}{dz^2}\right) &+ (\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2)\\
&+ 2\left(\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz} + \frac{d\xi}{dx}\cdot\frac{d\zeta}{dz} + \frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)\Bigg), \qquad (51)
\end{aligned}$$

which will give the corresponding equations,

$$\begin{aligned}
\text{א} &= \mathrm{N}\left(3\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) & \text{ל} &= \mathrm{N}\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\\
\text{ב} &= \mathrm{N}\left(3\frac{d\eta}{dy} + \frac{d\zeta}{dz} + \frac{d\xi}{dx}\right) & \text{ט} &= \mathrm{N}\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right) \qquad (52)\\
\text{ג} &= \mathrm{N}\left(3\frac{d\zeta}{dz} + \frac{d\xi}{dx} + \frac{d\eta}{dy}\right) & \text{כ} &= \mathrm{N}\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)
\end{aligned}$$

which will be found useful in discussing the conditions at the limits for this particular case. Previously, however, to examining the limits, it is necessary to consider the laws of propagation; and here I shall confine my attention to the particular integral for plane waves already used. In the present case equations (29) become

$$\begin{aligned} P &= N(3x^2+y^2+z^2), & P' &= N(2l^2+1), \\ Q &= N(3y^2+x^2+z^2), & Q' &= N(2m^2+1), \\ R &= N(3z^2+x^2+y^2), & R' &= N(2n^2+1), \\ F &= 2Nyz, & F' &= 2Nmn, \\ G &= 2Nxz, & G' &= 2Nln, \\ H &= 2Nxy, & H' &= 2Nlm. \end{aligned} \tag{53}$$

Substituting these values in (28), we obtain

$$\begin{aligned} \epsilon v^2\cos\alpha &= N((2l^2+1)\cos\alpha + 2lm\cos\beta + 2ln\cos\gamma), \\ \epsilon v^2\cos\beta &= N((2m^2+1)\cos\beta + 2mn\cos\gamma + 2lm\cos\alpha), \\ \epsilon v^2\cos\gamma &= N((2n^2+1)\cos\gamma + 2ln\cos\alpha + 2mn\cos\beta); \end{aligned} \tag{54}$$

from which we deduce, by multiplying by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, respectively, and adding

$$\epsilon v^2 = N(1 + 2(l\cos\alpha + m\cos\beta + n\cos\gamma)^2). \tag{55}$$

It is evident, on inspection, that $\cos\alpha = l$, $\cos\beta = m$, $\cos\gamma = n$, will satisfy equations (54), and consequently that one set of vibrations are normal to the wave-plane, and the other two in the wave-plane. The equation (55) will give for the two cases respectively,

$$\epsilon v^2 = 3N,$$
$$\epsilon v^2 = N,$$

the greater velocity belonging to the normal vibration.

These results may also be deduced from the surface of wave-slowness (31), for, by substituting in this equation the values (53), and expanding, it becomes

$$[3N(x^2+y^2+z^2)-1]\,[N(x^2+y^2+z^2)-1]^2 = 0, \tag{56}$$

which shows that it is composed of two concentric spheres, whose radii are $\frac{1}{\sqrt{N}}$, $\frac{1}{\sqrt{3N}}$, which, therefore, represent the *wave-slownesses* of the corresponding vibrations.

Let the case considered be that of two indefinitely extended homogeneous and uncrystalline solids, separated by a plane (which is assumed the plane $x, y$), then the six conditions to be satisfied at the limits are, as in (46),

$$\xi' = \xi'', \qquad \eta' = \eta'', \qquad \zeta' = \zeta'',$$
$$\text{ג}' = \text{ג}'', \qquad \text{ל}' = \text{ל}'', \qquad \text{מ}' = \text{מ}'',$$

which become, by means of equations (52),

$$\begin{aligned}
&\xi' = \xi'', \qquad \eta' = \eta'', \qquad \zeta' = \zeta'', \\
&\text{N}'\left(3\frac{d\zeta'}{dz} + \frac{d\xi'}{dx} + \frac{d\eta'}{dy}\right) = \text{N}''\left(3\frac{d\zeta''}{dz} + \frac{d\xi''}{dx} + \frac{d\eta''}{dy}\right), \\
&\text{N}'\left(\frac{d\eta'}{dz} + \frac{d\zeta'}{dy}\right) = \text{N}''\left(\frac{d\eta''}{dz} + \frac{d\zeta''}{dy}\right), \\
&\text{N}'\left(\frac{d\zeta'}{dx} + \frac{d\xi'}{dz}\right) = \text{N}''\left(\frac{d\zeta''}{dx} + \frac{d\xi''}{dz}\right).
\end{aligned} \tag{57}$$

The incident vibration may be either transversal or normal; but as the transverse vibration may be resolved into two vibrations, one perpendicular to, and the other in the plane of incidence, we must finally, in order to include all the possible varieties, discuss the three following cases :

1. Incident vibration transversal, and perpendicular to plane of incidence.
2. Incident vibration transversal, and in the plane of incidence.
3. Incident vibration normal, and in the plane of incidence.

The plane of separation is the plane of $x, y$ ; $z$ being perpendicular to this plane, and the intersection of the plane of incidence with the separating plane being the axis of $x$.

1st. The incident vibration transversal, and perpendicular to the plane of incidence. In this case we shall have $\xi = 0$, $\zeta = 0$; and since the vibration is similar along the front of the wave-plane, we shall also have $\eta$ independent of $y$. The differential equations (50) will therefore be reduced to

$$\epsilon'\frac{d^2\eta'}{dt^2} = \text{N}'\left(\frac{d^2\eta'}{dz^2} + \frac{d^2\eta'}{dx^2}\right), \quad \epsilon''\frac{d^2\eta''}{dt^2} = \text{N}''\left(\frac{d^2\eta''}{dz^2} + \frac{d^2\eta''}{dx^2}\right); \tag{58}$$

and the equations (52) will become

$$\begin{aligned} \aleph &= 0, & \text{ל} &= \mathrm{N}\frac{d\eta}{dz}, \\ \beth &= 0, & \text{מ} &= 0, \\ \gimel &= 0, & \text{כ} &= \mathrm{N}\frac{d\eta}{dx}; \end{aligned} \tag{59}$$

from which it is evident that the equations of condition at the limits (57) will be reduced to the following,

$$\eta' = \eta'', \qquad \mathrm{N}'\frac{d\eta'}{dz} = \mathrm{N}''\frac{d\eta''}{dz}; \tag{60}$$

recollecting that the condition, $z = 0$, is implicitly supposed in these latter equations, as they belong only to the limits. The equations of condition at the limits are thus reduced to *two*, which is exactly the number required; for as all the motion is perpendicular to the plane of incidence, there will be no normal reflected or refracted wave, but only one reflected and one refracted wave, whose vibrations are transversal and perpendicular to the plane of incidence, and whose intensities are the *two* unknown quantities in the problem. The equations (59) also prove that the *first transversal* is zero, and that the *second* lies in the plane of incidence.

As there are two waves in the first body, and one in the second, we shall have

$$\eta' = \tau\cos\phi + \tau_{,}\cos\phi_{,}, \qquad \eta'' = \tau'\cos\phi';$$

$\tau$, $\tau_{,}$, $\tau'$ being the incident, reflected, and refracted vibration respectively, and

$$\phi = \frac{2\pi}{\lambda}(lx + nz - vt), \quad \phi_{,} = \frac{2\pi}{\lambda_{,}}(l_{,}x + n_{,}z - v_{,}t), \quad \phi' = \frac{2\pi}{\lambda'}(l'x + n'z - v't);$$

$l$, $\lambda$, $v$ belonging to the incident wave, $l_{,}$, $\lambda_{,}$, $v_{,}$ to the reflected, and $l'$, $\lambda'$, $v'$ to the refracted.

Substituting these values in (59), and proceeding as in (42), we obtain

$$\frac{2\pi}{\mathrm{T}^2}.f' = \mathrm{N}'\left(\frac{\tau n}{\lambda}\sin\phi + \frac{\tau_{,}n_{,}}{\lambda_{,}}\sin\phi_{,}\right), \qquad \frac{2\pi}{\mathrm{T}^2}.f'' = \mathrm{N}''\left(\frac{\tau' n'}{\lambda'}\sin\phi'\right),$$

$$\frac{2\pi}{\mathrm{T}^2}.h' = \mathrm{N}'\left(\frac{\tau l}{\lambda}\sin\phi + \frac{\tau_{,}l_{,}}{\lambda_{,}}\sin\phi_{,}\right), \qquad \frac{2\pi}{\mathrm{T}^2}.h'' = \mathrm{N}''\left(\frac{\tau' l'}{\lambda'}\sin\phi'\right);$$

but since $l = l_{,}$, $v = v_{,}$, $\lambda = \lambda_{,}$, $n = -n_{,}$, and since $\frac{l}{l'} = \frac{\lambda}{\lambda'} = \frac{v}{v'} = \frac{\sin\theta}{\sin\theta'}$, where

$\theta, \theta'$ are the angles of incidence and refraction, we shall have, when $z=0$, $\phi = \phi_{,} = \phi'$. Substituting therefore the results obtained above in the equations of condition, $\eta' = \eta''$, $f' = f''$, we shall deduce finally,

$$\tau + \tau_{,} = \tau',$$

$$\text{N}'(\tau - \tau_{,})\frac{n}{\lambda} = \text{N}''.\frac{\tau' n'}{\lambda'};$$

or, writing $\text{N}'' = \text{K}\text{N}'$, and substituting their values for $n, n', \lambda, \lambda'$,

$$\tau + \tau_{,} = \tau',$$

$$\tau - \tau_{,} = \tau'.\text{K}\frac{\tan\theta}{\tan\theta'}; \qquad (61)$$

whence we obtain

$$2\tau = \tau'\left(1 + \text{K}\frac{\tan\theta}{\tan\theta'}\right), \qquad 2\tau_{,} = \tau'\left(1 - \text{K}\frac{\tan\theta}{\tan\theta'}\right);$$

and finally, for the two unknown vibrations,

$$\tau_{,} = \tau.\frac{\tan\theta' - \text{K}\tan\theta}{\tan\theta' + \text{K}\tan\theta}, \qquad \tau' = \tau.\frac{2\tan\theta'}{\tan\theta' + \text{K}\tan\theta}. \qquad (62)$$

To determine the actual position and magnitude of the *second* transversal, we have evidently, for a single wave,

$$\frac{2\pi}{\text{T}^2}.f = \text{N}\tau.\frac{n\sin\phi}{\lambda},$$

$$\frac{2\pi}{\text{T}^2}.h = \text{N}\tau.\frac{l\sin\phi}{\lambda};$$

whence we obtain

$$\frac{h}{f} = \tan\theta, \quad \frac{2\pi}{\text{T}^2}.\sqrt{f^2 + h^2} = \text{N}.\frac{\tau}{\lambda}\sin\phi,$$

i. e. the second transversal lies in the plane $x$, $z$, or plane of incidence, and makes with the axis of $x$ an angle equal to the angle of incidence.

2nd. Incident vibration transversal, and in the plane of incidence. In this case all the motion will continue in the plane of incidence; we shall, therefore, have $\eta = 0$; and reasoning similar to that used in the former case will show that $\xi$ and $\zeta$ are independent of $y$. The differential equations of motion will therefore become

$$\epsilon' \frac{d^2\xi'}{dt^2} = \text{N}'\left(3\frac{d^2\xi'}{dx^2} + \frac{d^2\xi'}{dz^2} + 2\frac{d^2\zeta'}{dxdz}\right), \quad \epsilon'' \frac{d^2\xi''}{dt^2} = \text{N}''\left(3\frac{d^2\xi''}{dx^2} + \frac{d^2\xi''}{dz^2} + 2\frac{d^2\zeta''}{dxdz}\right),$$

$$\epsilon \frac{d^2\zeta'}{dt^2} = \text{N}'\left(3\frac{d^2\zeta'}{dz^2} + \frac{d^2\zeta'}{dx^2} + 2\frac{d^2\xi'}{dxdz}\right), \quad \epsilon'' \frac{d^2\zeta''}{dt^2} = \text{N}''\left(3\frac{d^2\zeta''}{dz^2} + \frac{d^2\zeta''}{dx^2} + 2\frac{d^2\xi''}{dxdz}\right). \tag{63}$$

Also, since $\frac{d\xi}{dx} + \frac{d\zeta}{dz} = 0$, in consequence of the vibration being transversal, we shall have the equations

$$\begin{aligned} \text{א} &= \frac{d^2p}{dt^2} = \text{N}\left(3\frac{d\xi}{dx} + \frac{d\zeta}{dz}\right), & \text{ל} &= \frac{d^2f}{dt^2} = 0, \\ \text{ב} &= \frac{d^2q}{dt^2} = 0, & \text{מ} &= \frac{d^2g}{dt^2} = \text{N}\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right), \\ \text{ג} &= \frac{d^2r}{dt^2} = \text{N}\left(3\frac{d\zeta}{dz} + \frac{d\xi}{dx}\right), & \text{כ} &= \frac{d^2h}{dt^2} = 0; \end{aligned} \tag{64}$$

which prove that the *first* transversal lies in the place of incidence, and that the *second* is perpendicular to it; and also that the equations of condition at the limits are the *four* following:

$$\begin{gathered} \xi' = \xi'', \qquad \zeta' = \zeta'', \\ \text{N}'\left(3\frac{d\zeta'}{dz} + \frac{d\xi'}{dx}\right) = \text{N}''\left(3\frac{d\zeta''}{dz} + \frac{d\xi''}{dx}\right), \\ \text{N}'\left(\frac{d\zeta'}{dx} + \frac{d\xi'}{dz}\right) = \text{N}''\left(\frac{d\zeta''}{dx} + \frac{d\xi''}{dz}\right), \end{gathered} \tag{65}$$

which are exactly equal in number to the unknown quantities, which are the *four intensities* of the two reflected and two refracted vibrations; one reflected and one refracted wave having its vibration normal to the wave plane, and the other two waves having their vibrations transversal. If the incident vibration had been normal, it is easy to see that the equations of motion would be the same as in the present case, and also that the equations of condition at the limits (65) would remain unaltered; but, in the case of normal vibration, the equation $\frac{d\xi}{dx} + \frac{d\zeta}{dz} = 0$ does not hold; the equations (64) will therefore be different, and become the following:

$$
\begin{aligned}
&\aleph = \mathrm{N}\left(3\frac{d\xi}{dx} + \frac{d\zeta}{dz}\right), && \text{ל} = 0, \\
&\text{ב} = \mathrm{N}\left(\frac{d\xi}{dx} + \frac{d\zeta}{dz}\right), && \text{מ} = \mathrm{N}\left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right), \\
&\text{ג} = \mathrm{N}\left(3\frac{d\zeta}{dz} + \frac{d\xi}{dx}\right), && \text{נ} = 0.
\end{aligned} \qquad (66)
$$

As the conditions at the limits express that the real vibration is preserved, together with the perpendicular component of the *first* transversal and the horizontal component of the *second* transversal; and as two of the four reflected and refracted vibrations are normal, and two transversal, it will be necessary, in order to express the conditions at the limits, to calculate the two transversals for both kinds of waves.

Substituting, therefore, in equations (64), for $\xi$ and $\zeta$, their values for a single wave,

$$\xi = \tau\cos a\cos\phi, \qquad \zeta = \tau\cos\gamma\cos\phi,$$

$$\phi = \frac{2\pi}{\lambda}(lx + nz - vt),$$

we obtain

$$
\begin{aligned}
\frac{2\pi}{\mathrm{T}^2}.p &= \mathrm{N}\tau\frac{2l\cos a}{\lambda}\sin\phi, \\
\frac{2\pi}{\mathrm{T}^2}.r &= \mathrm{N}\tau\frac{2n\cos\gamma}{\lambda}\sin\phi, \\
\frac{2\pi}{\mathrm{T}^2}.g &= \mathrm{N}\tau\frac{l\cos\gamma + n\cos a}{\lambda}\sin\phi;
\end{aligned} \qquad (67)
$$

and by a similar process from (66), remembering that now the vibrations are normal,

$$
\begin{aligned}
&\frac{2\pi}{\mathrm{T}^2}.p = \mathrm{N}\tau.\frac{1+2l^2}{\lambda}\sin\phi, \\
&\frac{2\pi}{\mathrm{T}^2}.q = \mathrm{N}\tau.\frac{1}{\lambda}\sin\phi, \qquad \frac{2\pi}{\mathrm{T}^2}g = \mathrm{N}\tau.\frac{2ln}{\lambda}\sin\phi. \\
&\frac{2\pi}{\mathrm{T}^2}r = \mathrm{N}\tau.\frac{1+2n^2}{\lambda}\sin\phi,
\end{aligned} \qquad (68)
$$

If now, as in the former case, we designate by $(\phi, \phi_{,}, \phi', \phi_{,,}, \phi'')$ the phases of the incident, transverse reflected, transverse refracted, normal reflected, and normal refracted waves; and mark in a similar manner the other quantities in the problem, we shall obtain for the equations of condition (65), the following, ($z = 0$, making $\phi = \phi_{,} = \phi' = \phi_{,,} = \phi''$, as before):

$$\begin{gathered}
\tau\cos a + \tau_{,}\cos a_{,} + \tau_{,,}\cos a_{,,} = \tau'\cos a' + \tau''\cos a'', \\
\tau\cos\gamma + \tau_{,}\cos\gamma_{,} + \tau_{,,}\cos\gamma_{,,} = \tau'\cos\gamma' + \tau''\cos\gamma'', \\
\tau\frac{2n\cos\gamma}{\lambda} + \tau_{,}\frac{2n_{,}\cos\gamma_{,}}{\lambda_{,}} + \tau_{,,}\frac{1+2n_{,,}^2}{\lambda_{,,}} = \kappa\left(\tau'\frac{2n'\cos\gamma'}{\lambda'} + \tau''\frac{1+2n''^2}{\lambda''}\right), \\
\tau\frac{l\cos\gamma + n\cos a}{\lambda} + \tau_{,}\frac{l_{,}\cos\gamma_{,} + n_{,}\cos a_{,}}{\lambda_{,}} + \tau_{,,}\frac{2l_{,,}n_{,,}}{\lambda_{,,}} = \kappa\left(\tau'\frac{l'\cos\gamma' + n'\cos a'}{\lambda'} + \tau''\frac{2l''n''}{\lambda''}\right);
\end{gathered} \tag{69}$$

or substituting for $l$, $n$, &c. . . . $a$, $\gamma$, &c., their values in terms of $(\theta, \theta_{,}, \theta', \theta_{,,}, \theta'')$, which are all known quantities, we shall have, finally,

$$\begin{gathered}
(\tau + \tau_{,})\cos\theta + \tau_{,,}\sin\theta_{,,} = \tau'\cos\theta' + \tau''\sin\theta'', \\
(\tau - \tau_{,})\sin\theta + \tau_{,,}\cos\theta_{,,} = \tau'\sin\theta' - \tau''\cos\theta'', \\
(\tau + \tau_{,})\frac{\sin 2\theta}{\lambda} - \tau_{,,}\frac{1+2\cos^2\theta_{,,}}{\lambda_{,,}} = \kappa\left(\tau'\frac{\sin 2\theta'}{\lambda'} - \tau''\frac{1+2\cos^2\theta''}{\lambda''}\right), \\
(\tau - \tau_{,})\frac{\cos 2\theta}{\lambda} - \tau_{,,}\frac{\sin 2\theta_{,,}}{\lambda_{,,}} = \kappa\left(\tau'\frac{\cos 2\theta'}{\lambda'} + \tau''\frac{\sin 2\theta''}{\lambda''}\right);
\end{gathered} \tag{70}$$

$\kappa$ being equal to $\frac{N''}{N'}$.

These four equations determine completely, in terms of known quantities, the four unknown vibrations $(\tau_{,}, \tau', \tau_{,,}, \tau'')$.

To determine the actual position and magnitude of the two transversals in the present case, we must use the equations (67), which give for the *first* transversal the equations

$$\frac{r}{p} = -1, \quad \frac{\pi\lambda}{T^2}.\sqrt{p^2 + r^2} = N\tau.\frac{\sin 2\theta}{\sqrt{2}}\sin\phi,$$

which show that it is situated in the bisection of the angle between $z$ and $-x$. And for the *second* transversal,

$$\frac{2\pi\lambda}{T^2}g = N\tau.\cos 2\theta\,\sin\phi.$$

The position and magnitude of the transversals which have been just determined, belong to the transversals of a wave whose *real* vibration takes place in its own plane; and the position and magnitude of the transversals belonging to a normal vibration might be determined in a manner precisely similar from equations (68).

3rd. Incident vibration normal, and in the plane of incidence. The differential equations for this case, and the conditions at the limits, are the equations (63) and (65). Representing, therefore, by $(\phi, \phi_{,}, \phi', \phi_{,,}, \phi'')$ the phases of the incident, reflected normal, refracted normal, reflected transverse, and refracted transverse waves, the conditions at the limits will become the four following, which determine, as before, the four unknown intensities:

$$\begin{aligned}
(\tau + \tau_{,})\sin\theta + \tau_{,,}\cos\theta_{,,} &= \tau'\sin\theta' + \tau''\cos\theta'', \\
(\tau - \tau_{,})\cos\theta + \tau_{,,}\sin\theta_{,,} &= \tau'\cos\theta' - \tau''\sin\theta'', \\
(\tau + \tau_{,})\frac{1 + 2\cos^2\theta}{\lambda} - \tau_{,,}\frac{\sin 2\theta_{,,}}{\lambda_{,,}} &= \kappa\left(\tau'\frac{1 + 2\cos^2\theta'}{\lambda'} - \tau''\frac{\sin 2\theta''}{\lambda''}\right), \qquad (71) \\
(\tau - \tau_{,})\frac{\sin 2\theta}{\lambda} - \tau_{,,}\frac{\cos 2\theta_{,,}}{\lambda_{,,}} &= \kappa\left(\tau'\frac{\sin 2\theta'}{\lambda'} + \tau''\frac{\cos 2\theta''}{\lambda''}\right).
\end{aligned}$$

The two transversals which I have made use of in this Paper, in investigating the conditions at the limits, are merely *conceivable* mathematical lines, and have no real existence in the physical problem: but they are useful to assist in a geometrical conception of what really happens at the limits. There is one real *physical* vibration, which is to be conceived as accompanied by two *mathematical* vibrations, which depend upon the real one—each of the three vibrations giving its own conditions at the limits; so that the two transversals, although only conceptions of the mind, are naturally suggested by the conditions of the problem, and serve to convey a graphic representation of it.